CHAPTER 11

Valuing Stock Options: The Black–Scholes Model

In the early 1970s Fischer Black, Myron Scholes, and Robert Merton made a major breakthrough in the pricing of stock options.[1] This involved the development of what has become known as the Black–Scholes model. This model has had a huge influence on the way in which traders price and hedge options. It has also been pivotal to the growth and success of financial engineering in the 1980s and 1990s. An acknowledgment of the importance of the model came in 1997 when Myron Scholes and Robert Merton were awarded the Nobel prize for economics. Sadly Fischer Black died in 1995. Otherwise he would undoubtedly also have been one of the recipients of this prize.

In this chapter we present the Black–Scholes model for valuing European call and put options on a non-dividend-paying stock and discuss the assumptions on which it is based. We also consider more fully than in previous chapters the meaning of volatility and show how volatility can be either estimated from historical data or implied from option prices. Toward the end of the chapter we review how the Black–Scholes results can be extended to deal with European call and put options on dividend-paying stocks.

11.1 ASSUMPTIONS ABOUT HOW STOCK PRICES EVOLVE

A stock option pricing model must make some assumptions about how stock prices evolve over time. If a stock price is $100 today, what is the probability distribution for the price in one day or in one week or in one year?

The assumption underlying the Black–Scholes model is that (in the absence of dividends) stock prices follow a *random walk*. Percentage changes in the stock price

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[1] See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (May–June 1973): 637–59; and R. C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4 (spring 1973): 141–83.

in a short period of time are normally distributed. Define

μ: Expected return on the stock
σ: Volatility of the stock price

The mean of the percentage change in time δt is $\mu\delta t$. The standard deviation of the percentage change is $\sigma\sqrt{\delta t}$. The assumption underlying Black–Scholes is therefore

$$\frac{\delta S}{S} \sim \phi(\mu\,\delta t, \sigma\sqrt{\delta t}) \tag{11.1}$$

where δS is the change in the stock price, S, in time δt, and $\phi(m, s)$ denotes a normal distribution with mean m and standard deviation s.

The Lognormal Distribution

It can be shown that the random-walk assumption implies that the stock price at any future time has a *lognormal* distribution. The general shape of a lognormal distribution is shown in Figure 11.1. It can be contrasted with the more familiar normal distribution in Figure 11.2. Whereas a variable with a normal distribution can take any positive or negative value, a lognormally distributed variable is restricted to being positive. A normal distribution is symmetrical; a lognormal distribution is skewed with the mean, median, and mode all different.

A variable with a lognormal distribution has the property that its natural logarithm is normally distributed. The Black–Scholes assumption for stock prices therefore implies that $\ln S_T$ is normal, where S_T is the stock price at a future time T. The mean and standard deviation of $\ln S_T$ can be shown to be

$$\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T$$

and

$$\sigma\sqrt{T}$$

where S_0 is the current stock price. We can write this result as

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right] \tag{11.2}$$

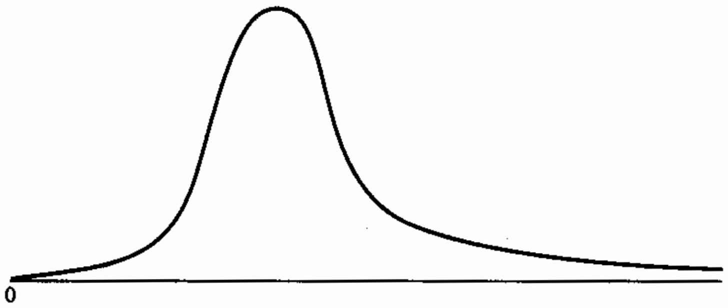

Figure 11.1 A lognormal distribution

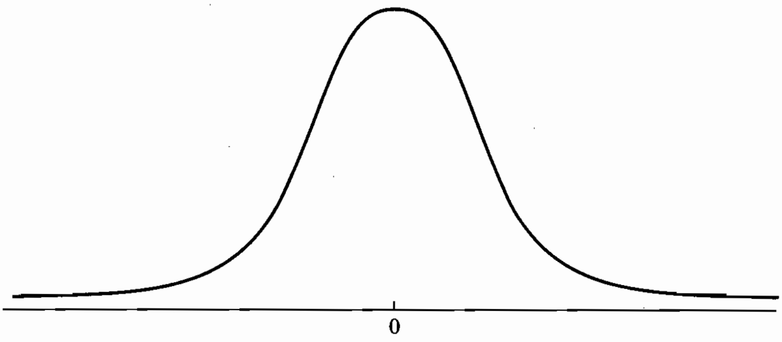

Figure 11.2 A normal distribution

The expected value or mean value of S_T, $E(S_T)$, is given by

$$E(S_T) = S_0 e^{\mu T} \tag{11.3}$$

This fits in with the definition of μ as the expected return. The variance of S_T, $\text{var}(S_T)$, can be shown to be given by

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

Example

Consider a stock with an initial price of \$40, an expected return of 16% per annum, and a volatility of 20% per annum. From equation (11.2), the probability distribution of the stock price, S_T, in six months is given by

$$\ln S_T \sim \phi\left[\ln 40 + \left(0.16 - \frac{0.2^2}{2}\right)0.5,\ 0.2\sqrt{0.5}\right]$$

or

$$\ln S_T \sim \phi(3.759,\ 0.141)$$

There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. Hence, with 95% confidence,

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

This implies

$$e^{3.759-1.96\times0.141} < S_T < e^{3.759+1.96\times0.141}$$

or

$$32.55 < S_T < 56.56$$

Thus, there is a 95% probability that the stock price in six months will lie between 32.55 and 56.56. The mean and variance of S_T are

$$40e^{0.16\times0.5} = 43.33$$

and

$$40^2 e^{2\times0.16\times0.5}(e^{0.2\times0.2\times0.5} - 1) = 37.93$$

From equation (11.2), it can be shown that

$$\ln\frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right] \tag{11.4}$$

When $T = 1$, the expression $\ln(S_T/S_0)$ is the continuously compounded return provided by the stock in one year.[2] The mean and standard deviation of the continuously compounded return in one year are therefore $\mu - \sigma^2/2$ and σ, respectively.

Example

> Consider a stock with an expected return of 17% per annum and a volatility of 20% per annum. The probability distribution for the rate of return (continuously compounded) realized over one year is normal, with mean
>
> $$0.17 - \frac{0.2^2}{2} = 0.15$$
>
> or 15% and standard deviation 20%. Because there is a 95% chance that a normally distributed variable will lie within 1.96 standard deviations of its mean, we can be 95% confident that the return realized over one year will be between −24.2% and +54.2%.

We now consider in more detail the nature of the expected return and volatility parameter in the lognormal stock price model.

11.2 EXPECTED RETURN

The expected return, μ, required by investors from a stock depends on the riskiness of the stock. The higher the risk, the higher the return. It also depends on the level of interest rates in the economy. The higher the risk-free interest rate, the higher the expected return required on any given stock. Fortunately, we do not have to concern ourselves with the determinants of μ in any detail. It turns out that the value of a stock option, when expressed in terms of the value of the underlying stock, does not depend on μ at all. Nevertheless, there is one aspect of the expected return from a stock that frequently causes confusion and is worth explaining.

Equation (11.1) shows that $\mu\,\delta t$ is the expected percentage change in the stock price in a very short period of time, δt. This means that μ is the expected return in a very short period of time δt. This return is expressed with a compounding period of δt and in the limit as δt tends to zero is continuously compounded. It is natural to assume that μ is also the expected continuously compounded return on the stock over a relatively long period of time. However, this is not the case. The continuously compounded return realized over T years is

$$\frac{1}{T}\ln\frac{S_T}{S_0}$$

and equation (11.4) shows that the expected value of this is $\mu - \sigma^2/2$.

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[2] As discussed in Chapter 3, it is important to distinguish between the continuously compounded return and the return with annual compounding. The former is $\ln(S_T/S_0)$. The latter is $(S_T - S_0)/S_0$.

The reason for the distinction between the μ in equation (11.1) and the $\mu - \sigma^2/2$ in equation (11.4) is subtle, but important. We start with equation (11.3)

$$E(S_T) = S_0 e^{\mu T}$$

Taking logarithms we get

$$\ln[E(S_T)] = \ln(S_0) + \mu T$$

It is now tempting to set $\ln[E(S_T)] = E[\ln(S_T)]$ so that $E[\ln(S_T)] - \ln(S_0) = \mu T$ or $E[\ln(S_T/S_0)] = \mu T$. However, we cannot do this because ln is a nonlinear function. In fact $\ln[E(S_T)] > E[\ln(S_T)]$ so that $E[\ln(S_T/S_0)] < \mu T$. This is consistent with equation (11.4).

Suppose we consider a very large number of very short periods of time of length δt. Define S_i as the stock price at the end of the ith interval and δS_i as $S_{i+1} - S_i$. Under the assumptions we are making for stock price behavior, the average of the returns on the stock in each interval is close to μ. In other words, μ is close to the arithmetic mean of the $\delta S_i/S_i$. However, the expected return over the whole period covered by the data, expressed with a compounding period of δt, is close to $\mu - \sigma^2/2$, not μ.[3] The following simple example illustrates what is going on.

Example

Suppose that the following is a sequence of returns per annum on a stock, measured using annual compounding:

15%, 20%, 30%, −20%, 25%

The arithmetic mean of the returns, calculated by taking the sum of the returns and dividing by 5, is 14%. However, an investor would actually earn less than 14% per annum by leaving the money invested in the stock for five years. The dollar value of \$100 at the end of the five years would be

$$100 \times 1.15 \times 1.20 \times 1.30 \times 0.80 \times 1.25 = \$179.40$$

By contrast, a 14% return with annual compounding would give

$$100 \times 1.14^5 = \$192.54$$

The actual average return earned by the investor, with annual compounding, is

$$(1.7940)^{1/5} - 1 = 0.124$$

or 12.4% per annum.

The arguments in this section show that the term *expected return* is ambiguous. It can refer either to μ or to $\mu - \sigma^2/2$. Unless otherwise stated, it will be used to refer to μ throughout this book.

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[3] If we define the *gross return* as one plus the regular return, the gross return over the whole period covered by the data is the geometric average of the gross returns in each time interval of length δt—not the arithmetic average. The geometric average of a set of numbers is always less than the arithmetic average unless the numbers happen to be equal.

11.3 VOLATILITY

The volatility of a stock, σ, is a measure of our uncertainty about the returns provided by the stock. "Old economy" stocks typically have a volatility between 20% and 40%. "New economy" stocks typically have a volatility between 40% and 60%.

From equation (11.4), the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in one year when the return is expressed using continuous compounding. From equation (11.2), $\sigma\sqrt{T}$ is the standard deviation of $\ln S_T$.

When T is small, equation (11.1) shows that $\sigma\sqrt{T}$ is approximately equal to the standard deviation of the percentage change in the stock price in time T. Suppose that $\sigma = 0.3$, or 30% per annum, and the current stock price is \$50. The standard deviation of the percentage change in the stock price in one week is approximately

$$30 \times \sqrt{\frac{1}{52}} = 4.16\%$$

A one-standard-deviation move in the stock price in one week is therefore 50×0.0416 or \$2.08.

Equation (11.1) shows that our uncertainty about a future stock price, as measured by its standard deviation, increases—at least approximately—with the square root of how far ahead we are looking. For example, the standard deviation of the stock price in four weeks is approximately twice the standard deviation in one week.

11.4 ESTIMATING VOLATILITY FROM HISTORICAL DATA

A record of stock price movements can be used to estimate volatility. The stock price is usually observed at fixed intervals of time (e.g., every day, every week, or every month). We define

$n+1$: Number of observations
S_i: Stock price at end of ith interval $(i = 0, 1, \dots, n)$
τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

An estimate, s, of the standard deviation of the u_i's is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2}$$

where $\bar{u}$ is the mean of the u_i's.

From equation (11.4), the standard deviation of the u_i's is $\sigma\sqrt{\tau}$. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$. It follows that σ itself can be estimated as $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The standard error of this estimate can be shown to be approximately $\hat{\sigma}/\sqrt{2n}$.

Choosing an appropriate value for n is not easy. More data generally lead to more accuracy, but σ does change over time and data that are too old may not be relevant for predicting the future. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. An often-used rule of thumb is to set n equal to the number of days to which the volatility is to be applied. Thus, if the volatility is to be used to value a two-year option, two years of daily historical data are used.

There is an important issue concerned with whether time should be measured in calendar days or trading days when volatility parameters are being estimated and used. This question will be discussed more fully later in this chapter.

Example

Table 11.1 shows a possible sequence of stock prices during 21 consecutive trading days. In this case

$$\sum u_i = 0.09531 \quad \text{and} \quad \sum u_i^2 = 0.00326$$

and the estimate of the standard deviation of the daily return is

$$\sqrt{\frac{0.00326}{19} - \frac{0.09531^2}{380}} = 0.01216$$

or 1.216%. Assuming that there are 252 trading days per year, $\tau = 1/252$ and the data give an estimate for the volatility per annum of $0.01216\sqrt{252} = 0.193$ or 19.3%. The standard error of this estimate is

$$\frac{0.193}{\sqrt{2 \times 20}} = 0.031$$

or 3.1% per annum.

The foregoing analysis assumes that the stock pays no dividends. It can be adapted to accommodate dividend-paying stocks. The return, u_i, during a time interval that includes an ex-dividend day is given by

$$u_i = \ln \frac{S_i + D}{S_{i-1}}$$

where D is the amount of the dividend. The return in other time intervals is still

$$u_i = \ln \frac{S_i}{S_{i-1}}$$

Because tax factors play a part in determining returns around an ex-dividend date, it is

Table 11.1 Computation of volatility

Day	Closing stock price (dollars)	Price relative S_i/S_{i-1}	Daily return $u_i = \ln(S_i/S_{i-1})$
0	20.00		
1	20.10	1.00500	0.00499
2	19.90	0.99005	−0.01000
3	20.00	1.00503	0.00501
4	20.50	1.02500	0.02469
5	20.25	0.98780	−0.01227
6	20.90	1.03210	0.03159
7	20.90	1.00000	0.00000
8	20.90	1.00000	0.00000
9	20.75	0.99282	−0.00720
10	20.75	1.00000	0.00000
11	21.00	1.01205	0.01198
12	21.10	1.00476	0.00475
13	20.90	0.99052	−0.00952
14	20.90	1.00000	0.00000
15	21.25	1.01675	0.01661
16	21.40	1.00706	0.00703
17	21.40	1.00000	0.00000
18	21.25	0.99299	−0.00703
19	21.75	1.02353	0.02326
20	22.00	1.01149	0.01143

probably best to discard altogether data for intervals that include an ex-dividend date when daily or weekly data is used.

11.5 ASSUMPTIONS UNDERLYING BLACK–SCHOLES

The assumptions made by Black and Scholes when they derived their option pricing formula were as follows:

1. Stock price behavior corresponds to the lognormal model (developed earlier in this chapter) with μ and σ constant.
2. There are no transactions costs or taxes. All securities are perfectly divisible.
3. There are no dividends on the stock during the life of the option.
4. There are no riskless arbitrage opportunities.
5. Security trading is continuous.
6. Investors can borrow or lend at the same risk-free rate of interest.
7. The short-term risk-free rate of interest, r, is constant.

Some of these assumptions have been relaxed by other researchers. For example, variations on the Black–Scholes formula can be used when r and σ are functions of time, and as we will see later in this chapter the formula can be adjusted to take dividends into account.

11.6 THE BLACK–SCHOLES/MERTON ANALYSIS

The Black–Scholes/Merton analysis is analogous to the no-arbitrage analysis used in Chapter 10 to value options when stock price changes are binomial. A riskless portfolio consisting of a position in the option and a position in the underlying stock is set up. In the absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This results in a differential equation that must be satisfied by the option.

The reason a riskless portfolio can be set up is that the stock price and the option price are both affected by the same underlying source of uncertainty: stock price movements. In any short period of time, the price of a call option is perfectly positively correlated with the price of the underlying stock; the price of a put option is perfectly negatively correlated with the price of the underlying stock. In both cases, when an appropriate portfolio of the stock and the option is set up, the gain or loss from the stock position always offsets the gain or loss from the option position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

Suppose, for example, that at a particular point in time the relationship between a small change in the stock price, δS, and the resultant small change in the price of a European call option, δc, is given by

$$\delta c = 0.4\,\delta S$$

This means that the slope of the line representing the relationship between c and S is 0.4, as indicated in Figure 11.3. The riskless portfolio would consist of

1. A long position in 0.4 share
2. A short position in 1 call option

There is one important difference between the Black–Scholes/Merton analysis and the analysis using a binomial model in Chapter 10. In Black–Scholes/Merton, the position that is set up is riskless for only a very short period of time. (Theoretically, it remains riskless only for an instantaneously short period of time.) To remain riskless, it must be frequently adjusted or *rebalanced.*[4] For example, the relationship between δc and δS might change from $\delta c = 0.4\,\delta S$ today to $\delta c = 0.5\,\delta S$ in two weeks. (If so, an extra 0.1 shares must be purchased for each call option sold to maintain a riskless portfolio.) It is

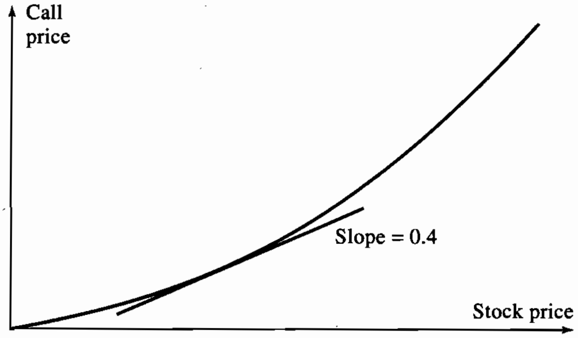

Figure 11.3 Relationship between c and S

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[4] We will examine the rebalancing of portfolios in more detail in Chapter 15.

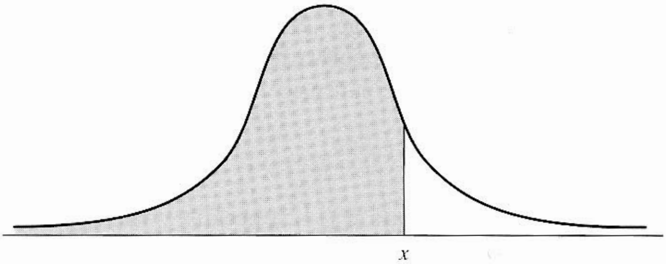

Figure 11.4 Shaded area represents $N(x)$

nevertheless true that the return from the riskless portfolio in any short period of time must be the risk-free interest rate. This is the key element in the Black–Scholes/Merton arguments and leads to their pricing formulas.

The Pricing Formulas

The Black–Scholes formulas for the prices of European calls and puts on non-dividend-paying stocks are[5]

$$c = S_0N(d_1) - Xe^{-rT}N(d_2) \tag{11.5}$$

$$p = Xe^{-rT}N(-d_2) - S_0N(-d_1) \tag{11.6}$$

where

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The function $N(x)$ is the cumulative probability function for a standardized normal variable. In other words, it is the probability that a variable with a standard normal distribution, $\phi(0, 1)$, will be less than x. It is illustrated in Figure 11.4. The remaining notation in equations (11.5) and (11.6) should be familiar. The variables c and p are the European call and put prices, S_0 is the stock price, X is the strike price, r is the risk-free interest rate (expressed with continuous compounding), T is the time to expiration, and σ is the volatility of the stock price. Because the American call price, C, equals the European call price, c, for a non-dividend-paying stock, equation (11.5) also gives the price of an American call. Unfortunately, no exact analytic formula for the value of an American put on a non-dividend-paying stock has been produced. We will look at numerical procedures that can be used to price American options in Chapter 17.

In theory, the Black–Scholes formula is correct only if the short-term interest rate, r, is constant. In practice, the formula is usually used with the interest rate, r, being set equal to the risk-free interest rate on an investment that lasts for time T.

Properties of the Black–Scholes Formulas

A full proof of the Black–Scholes formulas is beyond the scope of this book. At this stage we show that the formulas have the right general properties by considering what happens when some of the parameters take extreme values.

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[5] The software that accompanies this text can be used to carry out Black–Scholes calculations for options on stocks, currencies, indices, and futures contracts.

When the stock price, S_0, becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price X. From equation (3.9) we therefore expect the call price to be

$$S_0 - Xe^{-rT}$$

This is, in fact, the call price given by equation (11.5) because, when S_0 becomes very large, both d_1 and d_2 become very large, and $N(d_1)$ and $N(d_2)$ are both close to 1.0.

When the stock price becomes very large, the price of a European put option, p, approaches zero. This result is consistent with equation (11.6) because $N(-d_1)$ and $N(-d_2)$ are both close to zero when S_0 is large.

When the stock price becomes very small, both d_1 and d_2 become very large and negative. $N(d_1)$ and $N(d_2)$ are then both very close to zero, and equation (11.5) gives a price close to zero for the call option. This is as expected. Also, $N(-d_1)$ and $N(-d_2)$ become close to 1, so that the price of the put option given by equation (11.6) is close to $Xe^{-rT} - S_0$. This is also as expected.

The Cumulative Normal Distribution Function

The only problem in applying equations (11.5) and (11.6) is the computation of the cumulative normal distribution function, N. Tables for N are provided at the end of this book. The function can also be evaluated using a polynomial approximation. One such approximation that gives six-decimal-place accuracy is

$$N(x) = \begin{cases} 1 - N'(x)(a_1k + a_2k^2 + a_3k^3 + a_4k^4 + a_5k^5) & \text{when } x \geqslant 0 \\ 1 - N(-x) & \text{when } x < 0 \end{cases}$$

where

$$k = \frac{1}{1 + \gamma x}$$

$$\gamma = 0.2316419$$

$$a_1 = 0.319381530$$

$$a_2 = -0.356563782$$

$$a_3 = 1.781477937$$

$$a_4 = -1.821255978$$

$$a_5 = 1.330274429$$

and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Example

The stock price six months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. This means that $S_0 = 42$, $X = 40$, $r = 0.1$, $\sigma = 0.2$,

$T = 0.5$,

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278$$

and

$$Xe^{-rT} = 40e^{-0.1\times0.5} = 38.049$$

Hence, if the option is a European call, its value, c, is given by

$$c = 42N(0.7693) - 38.049N(0.6278)$$

If the option is a European put, its value, p, is given by

$$p = 38.049N(-0.6278) - 42N(-0.7693)$$

Using the polynomial approximation just given or the tables at the end of the book, we get

$$N(0.7693) = 0.7791, \qquad N(-0.7693) = 0.2209$$
$$N(0.6278) = 0.7349, \qquad N(-0.6278) = 0.2651$$

so that

$$c = 4.76, \qquad p = 0.81$$

Ignoring the time value of money, the stock price has to rise by \$2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by \$2.81 for the purchaser of the put to break even.

11.7 RISK-NEUTRAL VALUATION

A very important result in the pricing of derivatives is known as risk-neutral valuation. The principle was introduced in Chapter 10 and can be stated as follows:

> Any security dependent on other traded securities can be valued on the assumption that investors are risk neutral.

Note that risk-neutral valuation does not state that investors are risk neutral. What it does state is that derivatives such as options can be valued on the assumption that investors are risk neutral. It means that investors' risk preferences have no effect on the value of a stock option when it is expressed as a function of the price of the underlying stock. It explains why equations (11.5) and (11.6) do not involve the stock's expected return, μ.

Risk-neutral valuation is a very powerful tool, because in a risk-neutral world two particularly simple results hold:

1. The expected return from all securities is the risk-free interest rate.
2. The risk-free interest rate is the appropriate discount rate to apply to any expected future cash flow.

Options and other derivatives that provide a payoff at one particular time can be valued using risk-neutral valuation. The procedure is as follows:

1. Assume that the expected return from the underlying asset is the risk-free interest rate r (i.e., assume $\mu = r$).
2. Calculate the expected payoff from the option at its maturity.
3. Discount the expected payoff at the risk-free interest rate.

Application to Forward Contracts

This procedure can be used to derive the Black–Scholes formulas, but the mathematics is fairly complicated and will not be presented here. Instead, as an illustration, the procedure will be used to value a forward contract on a non-dividend-paying stock. (This contract has already been valued in Chapter 3 using a different approach.) We will make the assumption that interest rates are constant and equal to r.

Consider a long forward contract that matures at time T with delivery price K. The value of the contract at maturity is

$$S_T - K$$

The expected value of S_T was shown earlier in this chapter to be $S_0 e^{\mu T}$. In a risk-neutral world it becomes $S_0 e^{rT}$. The expected payoff from the contract at maturity in a risk-neutral world is therefore

$$S_0 e^{rT} - K$$

Discounting at the risk-free rate r for time T gives the value, f, of the forward contract today as

$$f = e^{-rT}(S_0 e^{rT} - K) = S_0 - Ke^{-rT}$$

This is in agreement with the result in equation (3.9).

11.8 IMPLIED VOLATILITIES

The one parameter in the Black–Scholes pricing formulas that cannot be observed directly is the volatility of the stock price. Earlier in this chapter we saw how volatility can be estimated from a history of the stock price. We now show how to calculate what is known as an *implied volatility*. This is the volatility implied by an option price observed in the market.[6]

To illustrate the basic idea, suppose that the value of a call on a non-dividend-paying stock is 1.90 when $S_0 = 21$, $X = 20$, $r = 0.1$, and $T = 0.25$. The implied volatility is the value of σ, which when substituted into equation (11.5) gives $c = 1.90$. It is not possible to invert equation (11.5) so that σ is expressed as a function of S_0, X, r, T, and c, but an iterative search procedure can be used to find the implied σ. We could start by trying $\sigma = 0.20$. This gives a value of c equal to 1.76, which is too low. Because c is an increasing function of σ, a higher value of σ is required. We could next try a value of 0.30 for σ. This gives a value of c equal to 2.10, which is too high, and means that σ must lie between 0.20 and 0.30. Next, we try a value of 0.25 for σ. This also proves to be too high, showing that σ lies between 0.20 and 0.25. Proceeding in this way, we can

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[6] Implied volatilities for European and American options on stocks, stock indices, foreign currencies, and futures can be calculated using the DerivaGem software supplied with this book.

halve the range for σ at each iteration and thereby calculate the correct value of σ to any required accuracy.[7] In this example, the implied volatility is 0.242, or 24.2% per annum.

Implied volatilities can be used to monitor the market's opinion about the volatility of a particular stock. Analysts often calculate implied volatilities from actively traded options on a certain stock and use them to calculate the price of a less actively traded option on the same stock. The way they do this is described in Chapter 14. It is important to note that the prices of deep-in-the-money and deep-out-of-the-money options are relatively insensitive to volatility. Implied volatilities calculated from these options therefore tend to be unreliable.

11.9 THE CAUSES OF VOLATILITY

We now move on to consider the causes of volatility. Some analysts claim that the volatility of a stock price is caused solely by the random arrival of new information about the future returns from the stock. Others maintain that volatility is caused largely by trading. An interesting question therefore is whether volatility is the same when the exchange is open as when it is closed.

Fama and French (see the end-of-chapter references) have tested this question empirically. They collected data on the stock price at the close of each trading day over a long period of time and then calculated:

1. The variance of stock price returns between the close of trading on one day and the close of trading on the next trading day when there are no intervening nontrading days
2. The variance of the stock price returns between the close of trading on Friday and the close of trading on Monday

If trading and nontrading days are equivalent, the variance in situation 2 should be three times as great as the variance in situation I. Fama found that it was only 22% higher. French's results were similar. He found that it was 19% higher.

These results suggest that volatility is far larger when the exchange is open than when it is closed. Proponents of the traditional view that volatility is caused only by new information might be tempted to argue that most new information on stocks arrives during trading hours.[8] However, studies of futures prices of agricultural commodities, which depend largely on the weather, have shown that they exhibit much the same behavior as stock prices; that is, they are much more volatile during trading hours. Presumably, news about the weather is equally likely to arise on any day. The only reasonable conclusion seems to be that volatility is to a large extent caused by trading itself.[9]

What are the implications of all this for the measurement of volatility and the Black–Scholes model? The results suggest that days when the exchange is closed should be ignored when volatilities are calculated from historical data and when volatilities are

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[7] This method is presented for illustration. Other, more powerful procedures are usually used in practice.

[8] In fact, this argument is questionable. Often, important announcements (e.g., those concerned with sales and earnings) are made when exchanges are closed.

[9] For a discussion, see the article by French and Roll referred to at the end of the chapter. We will consider one way in which trading can generate volatility when we look at portfolio insurance schemes in Chapter 15.

used to value options. For example, the volatility per annum should be calculated from the volatility per trading day using the formula

$$\text{Volatility per annum} = \text{Volatility per trading day} \times \sqrt{\text{Number of trading days per annum}}$$

This is the approach that was used earlier in this chapter in connection with the data in Table 11.1 and is the one usually used by traders. The number of trading days in a year is usually assumed to be 252 for stocks.

11.10 DIVIDENDS

Up to now we have assumed that the stock on which the option is written pays no dividends. In practice, this is not always the case. We now extend our results by assuming that the dividends paid on the stock during the life of an option can be predicted with certainty. When options last for relatively short periods of time (less than one year), the assumption is not unreasonable.

The date on which the dividend is paid should be assumed to be the ex-dividend date. On this date the stock price declines by the amount of the dividend.[10] The effect is to reduce the value of calls and increase the value of puts.

European Options

European options can be analyzed by assuming that the stock price is the sum of two components: a riskless component that will be used to pay the known dividends during the life of the option and a risky component. The riskless component at any given time is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate. The Black–Scholes formula is then correct if S_0 is set equal to the risky component. Operationally this means that the Black–Scholes formula can be used provided the stock price is reduced by the present value of all the dividends during the life of the option, the discounting being done from the ex-dividend dates at the risk-free rate. A dividend is included in the calculations only if its ex-dividend date occurs during the life of the option.

Example

Consider a European call option on a stock with ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be \$0.50. The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is

$$0.5e^{-0.09\times 2/12} + 0.5e^{-0.09\times 5/12} = 0.9741$$

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[10] For tax reasons the stock price may go down by somewhat less than the cash amount of the dividend. To take account of this phenomenon, we need to interpret the word *dividend* in the context of option pricing as the reduction in the stock price on the ex-dividend date caused by the dividend. Thus, if a dividend of \$1 per share is anticipated and the share price normally goes down by 80% of the dividend on the ex-dividend date, the dividend should be assumed to be \$0.80 for the purposes of the analysis.

The option price can therefore be calculated from the Black–Scholes formula with $S_0 = 39.0259$, $X = 40$, $r = 0.09$, $\sigma = 0.3$, and $T = 0.5$.

$$d_1 = \frac{\ln(39.0259/40) + (0.09 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2017$$

$$d_2 = \frac{\ln(39.0259/40) + (0.09 - 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = -0.0104$$

Using the polynomial approximation gives

$$N(d_1) = 0.5800 \quad \text{and} \quad N(d_2) = 0.4959$$

and from equation (11.5) the call price is

$$39.0259 \times 0.5800 - 40e^{-0.09\times0.5} \times 0.4959 = 3.67$$

or \$3.67.

With this procedure, σ in the Black–Scholes formula should be the volatility of the risky component of the stock price—not the volatility of the stock price itself. In practice, the two are often assumed to be the same. In theory, the volatility of the risky component is approximately $S_0/(S_0 - D)$ times the volatility of the stock price, where D is the present value of the dividends and S_0 is the stock price.

American Call Options

In Chapter 8 we saw that American call options should never be exercised early when the underlying stock pays no dividends. When dividends are paid, it is sometimes optimal to exercise at a time immediately before the stock goes ex-dividend. The reason is easy to understand. The dividend will make both the stock and the call option less valuable. If the dividend is sufficiently large and the call option is sufficiently in the money, it may be worth forgoing the remaining time value of the option in order to avoid the adverse effects of the dividend on the stock price.

In practice, call options are most likely to be exercised early immediately before the final ex-dividend date. The analysis in the Appendix at the end of the chapter indicates why this is so and derives the conditions under which early exercise can be optimal. Here we will describe an approximate procedure suggested by Fischer Black for valuing American calls on dividend-paying stocks.

Black's Approximation

Black's approximation involves calculating the prices of two European options:

1. An option that matures at the same time as the American option
2. An option maturing just before the latest ex-dividend date that occurs during the life of the option

The strike price, initial stock price, risk-free interest rate, and volatility are the same as for the option under consideration. The American option price is set equal to the higher of these two European option prices.

Example

Return to the previous example but suppose that the option is American rather than European. The present value of the first dividend is given by

$$0.5e^{-0.09\times 2/12} = 0.4926$$

The value of the option on the assumption that it expires just before the final ex-dividend date can be calculated using the Black–Scholes formula, with $S_0 = 39.5074$, $X = 40$, $r = 0.09$, $\sigma = 0.30$, and $T = 0.4167$. It is $3.52. Black's approximation involves taking the greater of this value and the value of the option when it can be exercised only at the end of six months. From the previous example, we know that the latter is $3.67. Black's approximation therefore gives the value of the American call as $3.67.

11.11 SUMMARY

The usual assumption in stock option pricing is that the price of a stock at some future time given its price today is lognormal. This in turn implies that the continuously compounded return from the stock in a period of time is normally distributed. Our uncertainty about future stock prices increases as we look further ahead. As a rough approximation, we can say that the standard deviation of the stock price is proportional to the square root of how far ahead we are looking.

To estimate the volatility, σ, of a stock price empirically, we need to observe the stock price at fixed intervals of time (e.g., every day, every week, or every month). For each time period the natural logarithm of the ratio of the stock price at the end of the time period to the stock price at the beginning of the time period is calculated. The volatility is estimated as the standard deviation of these numbers divided by the square root of the length of the time period in years. Usually days when the exchanges are closed are ignored in measuring time for the purposes of volatility calculations.

Stock option valuation involves setting up a riskless position in the option and the stock. Because the stock price and the option price both depend on the same underlying source of uncertainty, such a position can always be achieved. The position remains riskless for only a very short period of time. However, the return on a riskless position must always be the risk-free interest rate if there are to be no arbitrage opportunities. It is this fact that enables the option price to be valued in terms of the stock price. The original Black–Scholes equation gives the value of a European call or put option on a non-dividend-paying stock in terms of five variables: the stock price, the strike price, the risk-free interest rate, the volatility, and the time to expiration.

Surprisingly the expected return on the stock does not enter into the Black–Scholes equation. There is a general principle known as risk-neutral valuation, which states that any security dependent on other traded securities can be valued on the assumption that the world is risk neutral. The result proves to be very useful in practice. In a risk-neutral world the expected return from all securities is the risk-free interest rate, and the correct discount rate for expected cash flows is also the risk-free interest rate.

An implied volatility is the volatility that, when substituted into the Black–Scholes equation or its extensions, gives the market price of the option. Traders monitor implied volatilities and sometimes use the implied volatility from one stock option price to calculate the price of another option on the same stock. Empirical results show that the volatility of a stock is much higher when the exchange is open than when it is closed. This suggests that to some extent trading itself causes stock price volatility.

The Black–Scholes results can easily be extended to cover European call and put options on dividend-paying stocks. One procedure is to use the Black–Scholes formula with the stock price reduced by the present value of the dividends anticipated during the life of the option and the volatility equal to the volatility of the stock price net of the present value of these dividends. Fischer Black has suggested an approximate way of valuing American call options on a dividend-paying stock. His approach involves setting the price equal to the greater of two European option prices. The first European option expires at the same time as the American option; the second expires immediately prior to the final ex-dividend date.

Suggestions for Further Reading

On the Black–Scholes formula and its extensions

Black, F. "Fact and Fantasy in the Use of Options and Corporate Liabilities." *Financial Analysts Journal* 31 (July–August 1975): 36–41, 61–72.

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May–June 1973): 637–59.

Hull, J. *Options, Futures, and Other Derivatives.* 4th edn. Upper Saddle River, NJ: Prentice Hall, 2000.

Merton, R. C. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (spring 1973): 141–83.

Smith, C. W. "Option Pricing: A Review." *Journal of Financial Economics* 3 (March 1976): 3–51.

On the causes of volatility

Fama, E. E. "The Behavior of Stock Market Prices." *Journal of Business* 38 (January 1965): 34–105.

French, K. R. "Stock Returns and the Weekend Effect." *Journal of Financial Economics* 8 (March 1980): 55–69.

French, K., and R. Roll "Stock Return Variances: The Arrival of Information and the Reaction of Traders." *Journal of Financial Economics* 17 (September 1986): 5–26.

Quiz (Answers at End of Book)

11.1. What does the Black–Scholes stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the continuously compounded rate of return on the stock during the year?

11.2. The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

11.3. Explain how risk-neutral valuation could be used to derive the Black–Scholes formulas.

11.4. Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of $50 when the current stock price is $50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

11.5. What difference does it make to your calculations in the previous question if a dividend of \$1.50 is expected in two months?

11.6. What is meant by implied volatility? How would you calculate the volatility implied by a European put option price?

11.7. What is Black's approximation for valuing an American call option on a dividend-paying stock?

Questions and Problems (Answers in Solutions Manual)

11.8. A stock price is currently \$40. Assume that the expected return from the stock is 15% and its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a one-year period?

11.9. A stock price has an expected return of 16% and a volatility of 35%. The current price is \$38.

a. What is the probability that a European call option on the stock with an exercise price of \$40 and a maturity date in six months will be exercised?

b. What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

11.10. Prove that, with the notation in the chapter, a 95% confidence interval for S_T is between

$$S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

11.11. A portfolio manager announces that the average of the returns realized in each of the last 10 years is 20% per annum. In what respect is this statement misleading?

11.12. Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a derivative that pays off a dollar amount equal to

$$\frac{1}{T}\ln\left(\frac{S_T}{S_0}\right)$$

at time T. The variables S_0 and S_T denote the values of the stock price at time zero and time T.

a. Describe the payoff from this derivative.

b. Use risk-neutral valuation to calculate the price of the derivative at time zero.

11.13. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

11.14. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

11.15. A call option on a non-dividend-paying stock has a market price of \$2.50. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

11.16. Show that the Black–Scholes formula for a call option gives a price that tends to $\max(S_0 - X, 0)$ as $T \to 0$.

11.17. Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

11.18. Consider an American call option on a stock. The stock price is \$70, the time to maturity is eight months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after three months and again after six months. Use the results in the appendix to show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

11.19. A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with Black–Scholes?

	Maturity (months)		
Strike Price (\$)	3	6	12
45	7.00	8.30	10.50
50	3.50	5.20	7.50
55	1.60	2.90	5.10

11.20. Show that the Black–Scholes formulas for call and put options satisfy put–call parity.

11.21. Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than X?

Assignment Questions

11.22. A stock price is currently \$50. Assume that the expected return from the stock is 18% per annum and its volatility is 30% per annum. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine 95% confidence intervals.

1.23. Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2

Estimate the stock price volatility. What is the standard error of your estimate?

1.24. A financial institution plans to offer a derivative that pays off a dollar amount equal to S_T^2 at time T where S_T is the stock price at time T. Assume no dividends. Defining other variables as necessary use risk-neutral valuation to calculate the price of the derivative at time zero. (*Hint*: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in Section 11.1.)

1.25. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

a. What is the price of the option if it is a European call?
b. What is the price of the option if it is an American call?
c. What is the price of the option if it is a European put?
d. Verify that put–call parity holds.

11.26. Assume that the stock in Problem 11.25 is due to go ex-dividend in 1.5 months. The expected dividend is 50 cents.
a. What is the price of the option if it is a European call?
b. What is the price of the option if it is a European put?
c. Use the results in the Appendix to this chapter to determine whether there are any circumstances under which the option is exercised early.

11.27. Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends of 40 cents are expected during the life of the option, with ex-dividend dates at the end of two months and five months. Use Black's approximation and the DerivaGem software to value the option. Suppose now that the dividend is D of each ex-dividend date. Use the results in the Appendix to determine how high D can be without the American option being exercised early.

APPENDIX

The Early Exercise of American Call Options on Dividend-Paying Stocks

In Chapter 8 we saw that it is never optimal to exercise an American call option on a non-dividend-paying stock before the expiration date. A similar argument shows that the only times when a call option on a dividend-paying stock should be exercised are immediately before an ex-dividend date and on the expiration date. We assume that n ex-dividend dates are anticipated and that $t_1, t_2, \ldots, t_n$ are the moments in time immediately prior to the stock going ex-dividend, with $t_1 < t_2 < \cdots < t_n$. The dividends at these times will be denoted by $D_1, D_2, \ldots, D_n$, respectively.

We start by considering the possibility of early exercise immediately prior to the final ex-dividend date (i.e., at time t_n). If the option is exercised at time t_n, the investor receives

$$S(t_n) - X$$

If the option is not exercised, the stock price drops to $S(t_n) - D_n$. As shown in Chapter 8, a lower bound for the price of the option is then

$$S(t_n) - D_n - Xe^{-r(T-t_n)}$$

It follows that if

$$S(t_n) - D_n - Xe^{-r(T-t_n)} \geqslant S(t_n) - X$$

that is,

$$D_n \leqslant X(1 - e^{-r(T-t_n)}) \tag{11A.1}$$

it cannot be optimal to exercise at time t_n. On the other hand, if

$$D_n > X(1 - e^{-r(T-t_n)}) \tag{11A.2}$$

it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S(t_n)$. The inequality in equation (11A.2) is most likely to be satisfied when the final ex-dividend date is fairly close to the maturity of the option (i.e., when $T - t_n$ is small) and the dividend is large.

Consider next, time t_{n-1}, the penultimate ex-dividend date. If the option is exercised at time t_{n-1}, the investor receives

$$S(t_{n-1}) - X$$

If the option is not exercised at time t_{n-1}, the stock price drops to $S(t_{n-1}) - D_{n-1}$, and the earliest subsequent time at which exercise could take place is t_n. A lower bound to the option price if it is not exercised at time t_{n-1} is

$$S(t_{n-1}) - D_{n-1} - Xe^{-r(t_n - t_{n-1})}$$

It follows that if

$$S(t_{n-1}) - D_{n-1} - Xe^{-r(t_n - t_{n-1})} \geqslant S(t_{n-1}) - X$$

or

$$D_{n-1} \leqslant X(1 - e^{-r(t_n - t_{n-1})})$$

it is not optimal to exercise at time t_{n-1}. Similarly, for any $i < n$, if

$$D_i \leqslant X(1 - e^{-r(t_{i+1}-t_i)}) \tag{11A.3}$$

it is not optimal to exercise at time t_i.

The inequality in equation (11A.3) is approximately equivalent to

$$D_i \leqslant Xr(t_{i+1} - t_i)$$

Assuming that X is fairly close to the current stock price, the dividend yield on the stock would have to be either close to or above the risk-free rate of interest for the inequality not to be satisfied. This is not usually the case.

We can conclude from this analysis that, in most circumstances, the only time that needs to be considered for the early exercise of an American call is the final ex-dividend date, t_n. Furthermore, if the inequality in equation (11A.3) holds for $i = 1, 2, \dots, n - 1$ and the inequality in equation (11A.1) also holds, we can be certain that early exercise is never optimal.

Example

Consider the example that was used in this chapter to value European options on dividend-paying stocks: $S_0 = 40$, $X = 40$, $r = 0.09$, $\sigma = 0.30$, $t_1 = 0.1667$, $t_2 = 0.4167$, $T = 0.5$, and $D_1 = D_2 = 0.5$. We suppose that the option is American rather than European. In this case

$$X(1 - e^{-r(t_2-t_1)}) = 40(1 - e^{-0.09\times0.25}) = 0.89$$

Because this is greater than 0.5, it follows from equation (11A.3) that the option should never be exercised on the first ex-dividend date. Also

$$X(1 - e^{-r(T-t_2)}) = 40(1 - e^{-0.09\times0.08333}) = 0.30$$

Because this is less than 0.5, it follows from equation (11A.1) that when the option is sufficiently deeply in the money, it should be exercised on the second ex-dividend date.