CHAPTER 18

Interest Rate Options

Interest rate options are options whose payoffs are dependent in some way on the level of interest rates. In recent years they have become increasingly popular. Many different types of interest rate options now trade very actively both in the over-the-counter market and on exchanges. The chapter discusses some of the products and how they are used. It describes in some detail the standard market models that are used for pricing three popular over-the-counter instruments: European bond options, interest rate caps and floors, and European swap options. These models are in the spirit of the original Black–Scholes model for European stock options and are based on the assumption that a key market variable will be lognormally distributed at a future time.

18.1 EXCHANGE-TRADED INTEREST RATE OPTIONS

Among the most actively traded interest rate options offered by exchanges in the United States are those on Treasury bond futures, Treasury note futures, and Eurodollar futures. Table 13.3 in Chapter 13 shows the closing prices for these instruments on March 15, 2001.

A Treasury bond futures option is an option to enter a Treasury bond futures contract. As mentioned in Chapter 5, one Treasury bond futures contract is for the delivery of $100,000 of Treasury bonds. The price of a Treasury bond futures option is quoted as a percentage of the face value of the underlying Treasury bonds to the nearest sixty-fourth of 1%. Table 13.3 gives the price of the April call futures option on a Treasury bond on March 15, 2001, as 2-11, or $2\frac{11}{64}$ percent of the bond principal, when the strike price is 104. This means that one contract costs $2,171.87. The quotes for options on Treasury notes are similar.

An option on Eurodollar futures is an option to enter into a Eurodollar futures contract. As explained in Chapter 5, when the Eurodollar futures quote changes by one basis point, or 0.01, there is a gain or loss on a Eurodollar futures contract of $25. Similarly, in the pricing of options on Eurodollar futures, one basis point represents $25. In the shortest maturity contract, prices are quoted to the nearest quarter of a basis point. For the next two months they are quoted to the nearest half basis point. The *Wall Street Journal* quote for the CME Eurodollar futures contract in Table 13.3 should be multiplied by 10 to get the CME quote in basis points. The 5.92 quote for the CME March call futures option when the strike price is 94.50 in Table 13.3 should be interpreted as 5.925 and indicates that the CME quote is 59.25 basis points and one contract costs 59.25 × \$25 = \$1,481.25; the 10.30 quote for the April contract indicates that the CME quote is 103 basis points; and so on.

Interest rate futures option contracts work in the same way as the other futures options contracts discussed in Chapter 13. For example, the payoff from a call is $\max(F - X, 0)$, where F is the futures price at the time of exercise and X is the strike price. In addition to the cash payoff, the option holder obtains a long position in the futures contract when the option is exercised and the option writer obtains a corresponding short position.

Interest rate futures prices increase when bond prices increase (i.e., when interest rates fall). They decrease when bond prices decrease (i.e., when interest rates rise). An investor who thinks that short-term interest rates will rise can speculate by buying put options on Eurodollar futures, whereas an investor who thinks the rates will fall can speculate by buying call options on Eurodollar futures. An investor who thinks that long-term interest rates will rise can speculate by buying put options on Treasury note futures or Treasury bond futures, whereas an investor who thinks the rates will fall can speculate by buying call options on these instruments.

Example 1

Suppose that it is February and the futures price for the June Eurodollar contract is 93.82. (This corresponds to a three-month Eurodollar interest rate of 6.18% per annum.) The price of a call option on the contract with a strike price of \$94.00 is quoted at the CME as 20 basis points. This option could be attractive to an investor who feels that interest rates are likely to come down. Suppose that short-term interest rates do drop by about 100 basis points and the investor exercises the call when the Eurodollar futures price is 94.78. (This corresponds to a three-month Eurodollar interest rate of 5.22% per annum.) The payoff is 25 × (94.78 − 94.00) × 100 = \$1,950. The cost of the contract is 20 × 25 = \$500. The investor's profit is therefore \$1,450.

Example 2

Suppose that it is August and the futures price for the December Treasury bond contract traded on the CBOT is 96-09 (or $96\frac{9}{32} = 96.28125$). The yield on long-term government bonds is about 8.4% per annum. An investor who feels that this yield will fall by December might choose to buy December calls with a strike price of 98. Assume that the price of these calls is 1-04 (or $1\frac{4}{64} = 1.0625\%$ of the principal). If long-term rates fall to 8% per annum and the Treasury bond futures price rises to 100-00, the investor will make a net profit per \$100 of bond futures of

$$100.00 - 98.00 - 1.0625 = 0.9375$$

Because one option contract is for the purchase or sale of instruments with a face value of \$100,000, the investor would make a profit of \$937.50 per option contract bought.

18.2 EMBEDDED BOND OPTIONS

Some bonds contain embedded call and put options. For example, a *callable bond* contains provisions that allow the issuing firm to buy back the bond at a predetermined price at certain times in the future. The holder of such a bond has sold a call option to the issuer. The strike price or call price in the option is the predetermined price that must be paid by the issuer to the holder to buy back the bond. Callable bonds usually cannot be called for the first few years of their life. (This is known as a *lock out period*.) After that the call price is usually a decreasing function of time. For example, a 10-year callable bond might have no call privileges for the first two years. After that the issuer might have the right to buy the bond back at a price of $110.00 in years 3 and 4 of its life, at a price of $107.50 in years 5 and 6, at a price of $106.00 in years 7 and 8, and at a price of $103.00 in years 9 and 10. The value of the call option is reflected in the quoted yields on bonds. Bonds with call features generally offer higher yields than bonds with no call features.

A *puttable bond* contains provisions that allow the holder to demand early redemption at a predetermined price at certain times in the future. The holder of such a bond has purchased a put option on the bond as well as the bond itself. Because the put option increases the value of the bond to the holder, bonds with put features provide lower yields than bonds with no put features. A simple example of a puttable bond is a ten-year retractable bond in which the holder has the right to be repaid at the end of five years.

A number of instruments other than bonds have embedded interest rate options. Sometimes the options are bond options. For example, the early redemption privileges on fixed-rate deposits are a put option on a bond. The prepayment privileges on a fixed-rate loan are a call option on a bond. Also, a loan commitments made by a bank or other financial institution is a put option on a bond. Suppose, for example, that a bank quotes a five-year interest rate of 10% per annum to a potential borrower and states that the rate is good for the next two months. The client has in effect obtained the right to sell a five-year bond with a 10% coupon to the financial institution for its face value any time within the next two months.

18.3 BLACK'S MODEL

Since the Black–Scholes model was first published in 1973, it has become a very popular tool. As explained in Chapters 12 and 13, the model has been extended so that it can be used to value options on foreign exchange, options on indices, and options on futures contracts. As indicated in Chapter 14, traders have found flexible ways of using the model to reflect their beliefs. It is not surprising, therefore, that the model has been extended so that it covers interest rate derivatives.

The extension of the Black–Scholes model that is most widely used in the interest rate area is known as Black's model.[1] As discussed in Section 13.8, this was originally developed for valuing options on commodity futures. In this chapter we explain how it is used to value a number of different types of interest rate derivatives.

[1] See F. Black, "The Pricing of Commodity Contracts," *Journal of Financial Economics* 3 (March 1976): 167–79.

Using Black's Model to Price European Options

Consider a European call option on a variable V. We define:

- T: Time to maturity of the option
- F: Futures price of V for a contract maturing at time T
- F_0: Value of F at time zero
- F_T: Value of F at time T
- X: Strike price of the option
- r: Interest rate for maturity T
- σ: Volatility of F
- V_T: Value of V at time T

The option pays off $\max(V_T - X, 0)$ at time T. Because $F_T = V_T$, we can also regard the option as paying off $\max(F_T - X, 0)$ at time T. As shown in Chapter 13, Black's model gives the value, c, of the option at time zero as

$$c = e^{-rT}[F_0 N(d_1) - XN(d_2)] \tag{18.1}$$

where

$$d_1 = \frac{\ln(F_0/X) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/X) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The value, p, of the corresponding put option is given by

$$p = e^{-rT}[XN(-d_2) - F_0 N(-d_1)] \tag{18.2}$$

Extension of Black's Model

We can extend Black's model by allowing the time when the payoff is made to be different from T. Assume that the payoff on the option is calculated from the value of the variable V at time T, but that the payoff is delayed until time T^* where $T^* \geqslant T$. In this case it is necessary to discount the payoff from time T^* instead of from time T. We define r^* as the interest rate for maturity T^*, and equations (18.1) and (18.2) become

$$c = e^{-r^*T^*}[F_0 N(d_1) - XN(d_2)] \tag{18.3}$$

$$p = e^{-r^*T^*}[XN(-d_2) - F_0 N(-d_1)] \tag{18.4}$$

where

$$d_1 = \frac{\ln(F_0/X) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/X) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

How the Model Is Used

When Black's model is used to price European interest rate options trading in the over-the-counter market, the variable F_0 in equations (18.1) to (18.4) is usually set equal to the forward price of V rather than its futures price. Recall from Chapter 5 that futures prices and forward prices are equal when interest rates are constant, but different when they are stochastic. When we are dealing with an option on an interest-rate-dependent variable, the assumption that F_0 is a forward price is questionable. However, it turns out (at least for the products considered in this chapter) that this assumption exactly offsets another assumption that is usually made when Black's model is used. This is that interest rates are constant for the purposes of discounting. When used to value interest rate options, Black's model therefore has a stronger theoretical basis than is sometimes supposed.[2]

18.4 EUROPEAN BOND OPTIONS

A European bond option is an option to buy or sell a bond for a certain price, X, at a certain time, T. A common assumption in valuing bond options is that the bond price is lognormal at time T. Equations (18.1) and (18.2) can then be used with F_0 equal to the forward bond price. The variable σ is the volatility of F so that $\sigma\sqrt{T}$ is the standard deviation of the logarithm of the bond price at time T.

As explained in Chapter 5, F_0 can be calculated from today's spot bond price, B, using the formula

$$F_0 = (B - I)e^{rT} \tag{18.5}$$

where I is the present value of the coupons that will be paid during the life of the option and r is the interest rate for a maturity T. In this formula both the spot bond price and the forward bond price are cash prices rather than quoted prices. (The relationship between cash and quoted bond prices is explained in Chapter 5; the cash price is the quoted price plus accrued interest.) Traders refer to the quoted price of a bond as the "clean price" and the cash price as the "dirty price."

The strike price, X, in equations (18.1) and (18.2) should be the dirty (that is, cash) strike price. In the choice of the correct value for X, the precise terms of the option are therefore important. If the strike price is defined as the cash amount that is exchanged for the bond when the option is exercised, X should be put equal to this strike price. If the strike price is the clean price applicable when the option is exercised (as it is in most exchange-traded bond options), X should be set equal to the strike price plus accrued interest at the expiration date of the option.

Example

Consider a 10-month European call option on a 9.75-year bond with a face value of \$1,000. (When the option matures, the bond will have 8 years and 11 months remaining.) Suppose that the current cash bond price is \$960, the strike price is \$1,000, the 10-month risk-free interest rate is 10% per annum, and the forward bond price volatility is 9% per annum. The bond pays a semiannual coupon of 10%, and coupon payments of \$50 are expected in 3 months and 9 months. (This means that the accrued interest is \$25 and the quoted bond price is \$935.) We suppose that the 3-month and 9-month risk-free interest rates are 9.0% and 9.5%

[2] For an explanation of this, see J. C. Hull, *Options, Futures, and Other Derivatives*, 4th edn. (Upper Saddle River, N.J.: Prentice Hall, 2000), Chapter 19.

per annum, respectively. The present value of the coupon payments is therefore

$$50e^{-0.09\times0.25} + 50e^{-0.095\times0.75} = 95.45$$

or \$95.45. The bond forward price, from equation (18.5), is given by

$$F_0 = (960 - 95.45)e^{0.1\times10/12} = 939.68$$

(a) If the strike price is the cash price that would be paid for the bond on exercise, the parameters for equation (18.1) are $F_0 = 939.68$, $X = 1000$, $r = 0.1$, $\sigma = 0.09$, and $T = 0.8333$. The price of the call option is \$9.49.

(b) If the strike price is the quoted price that would be paid for the bond on exercise, one-month's accrued interest must be added to X, because the maturity of the option is one month after a coupon date. This produces a value for X of

$$1{,}000 + 50 \times 0.16667 = 1{,}008.33$$

The values for the other parameters in equation (18.1) are unchanged ($F_0 = 939.68$, $r = 0.1$, $\sigma = 0.09$, and $T = 0.8333$). The price of the option is \$7.97.

The volatility used in Black's model to value a bond option depends on both the life of the option and the life of the underlying bond. Figure 18.1 shows how the standard deviation of the logarithm of a bond's price changes with time. The standard deviation is zero today, because there is no uncertainty about the bond's price today. It is also zero at the bond's maturity, because we know that the bond's price will equal its face value at maturity. Between today and the maturity of the bond, the standard deviation first increases and then decreases. The volatility, σ, used in Black's model is

$$\frac{\text{Standard deviation of logarithm of bond price at maturity of option}}{\sqrt{\text{Time to maturity of option}}}$$

Figure 18.2 shows a typical pattern for σ as a function of the life of the option. In general σ declines as the life of the option increases. It also tends to be an increasing function of the life of the underlying bond when the life of the option is held fixed.

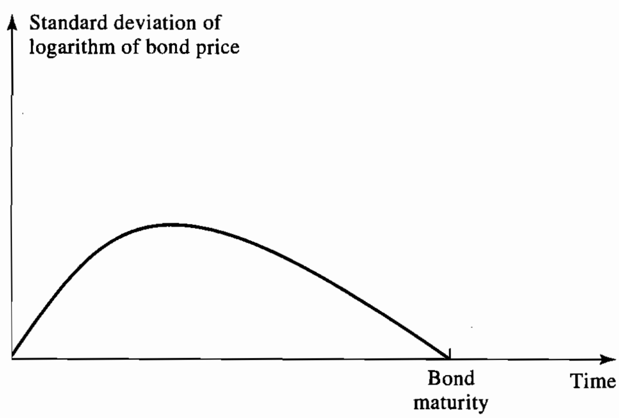

Figure 18.1 Standard deviation of logarithm of bond price as a function of time

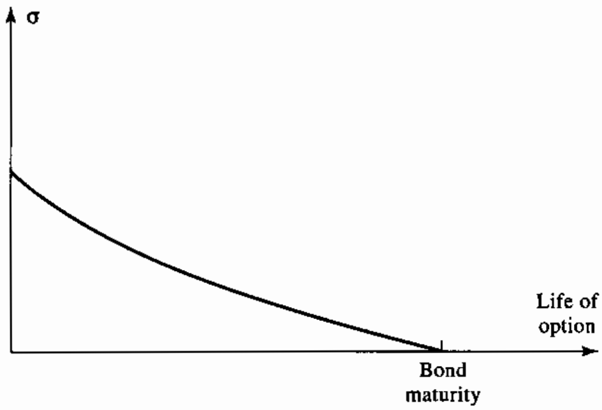

Figure 18.2 Variation of σ with life of bond option

Yield Volatilities

The volatilities that are quoted for bond options are often yield volatilities rather than price volatilities. The duration concept, introduced in Chapter 5, is used by the market to convert a quoted yield volatility into a price volatility. Suppose that D is the modified duration of the bond underlying the option at the option maturity, as defined in Chapter 5. The relationship between the change in the forward bond price, F, and its yield, y_F, at the maturity of the option is

$$\frac{\delta F}{F} \approx -D\,\delta y_F$$

or

$$\frac{\delta F}{F} \approx -D y_F \frac{\delta y_F}{y_F}$$

Volatility is a measure of the standard deviation of percentage changes in the value of a variable. This equation therefore suggests that the volatility of the forward bond price, σ, used in Black's model can be approximately related to the volatility of the forward bond yield, σ_y, by

$$\sigma = D y_0 \sigma_y \tag{18.6}$$

where y_0 is the initial value of y_F. When a yield volatility is quoted for a bond option, the implicit assumption is usually that it will be converted to a price volatility using equation (18.6), and that this volatility will then be used in conjunction with equation (18.1) or (18.2) to obtain a price. Suppose that the bond underlying a call option will have a modified duration of five years at option maturity, the forward yield is 8%, and the forward yield volatility quoted by a broker is 20%. This means that the market price of the option corresponding to the broker quote is the price given by equation (18.1) when the volatility variable, σ, is

$$5 \times 0.08 \times 0.2 = 0.08$$

or 8% per annum.

The Bond_Options worksheet of the software DerivaGem accompanying this book can be used to price European bond options using Black's model by selecting Black-

European as the Pricing Model. The user inputs a yield volatility, which is handled in the way just described. The strike price can be the cash or quoted strike price.

Example

Consider a European put option on a ten-year bond with a principal of 100. The coupon is 8% per year payable semiannually. The life of the option is 2.25 years and the strike price of the option is 115. The forward yield volatility is 20%. The zero curve is flat at 5% with continuous compounding. DerivaGem shows that the quoted price of the bond is 122.055. The price of the option when the strike price is a quoted price is 2.613. When the strike price is a cash price, the price of the option is $1.938. (Note that DerivaGem's prices may not exactly agree with manually calculated prices because DerivaGem assumes 365 days per year and rounds times to the nearest whole number of days.)

18.5 INTEREST RATE CAPS

A popular interest rate option offered by financial institutions in the over-the-counter market is an *interest rate cap*. Interest rate caps can best be understood by first considering a floating-rate note where the interest rate is reset periodically equal to LIBOR. The time between resets is known as the *tenor*. Suppose the tenor is three months. The interest rate on the note for the first three months is the initial three-month LIBOR rate; the interest rate for the next three months is set equal to the three-month LIBOR rate prevailing in the market in three months' time; and so on.

An interest rate cap is designed to provide insurance against the rate of interest on the floating-rate note rising above a certain level. This level is known as the *cap rate*. The operation of the cap is illustrated schematically in Figure 18.3. Suppose that the principal amount is $10 million, the life of the cap is five years, and the cap rate is 8%. (Because the tenor is three months, this cap rate is expressed with quarterly compounding.) Suppose that on a particular reset date the three-month LIBOR interest

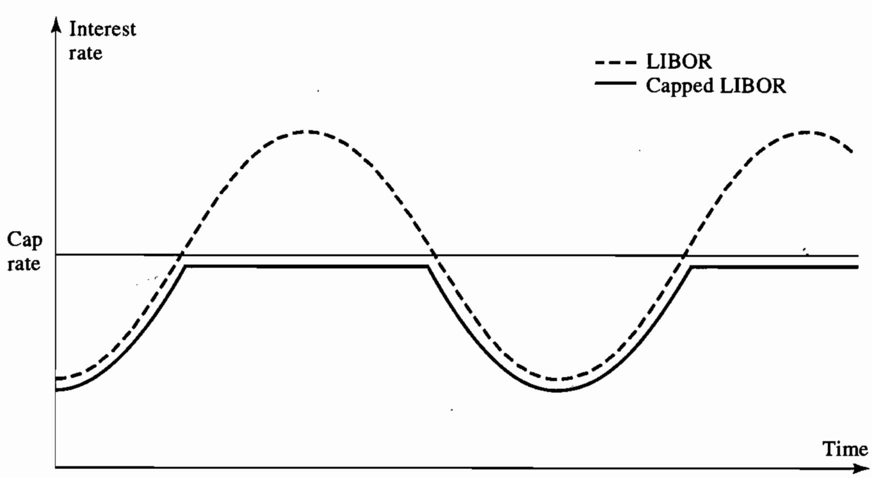

Figure 18.3 Effect of a cap in providing insurance against LIBOR rising above the cap rate

Table 18.1 Use of an interest rate cap

From the Trader's Desk

A company entering into a five-year $10 million floating-rate loan agreement is concerned about possible increases in interest rates. The rate on the loan is 3-month LIBOR plus 30 basis points. The company would like to buy protection against the rate rising above 8.3% per annum on any of the 19 reset dates.

The Strategy

The company buys from a financial institution a five-year interest rate cap with a cap rate of 8% per annum and a $10 million principal. The interest rate cap guarantees that whenever 3-month LIBOR proves to be greater than 8% per annum on a reset date, the financial institution will pay the difference between 3-month LIBOR and 8% per annum. The cap can be viewed as a portfolio of 19 call options on 3-month LIBOR.

rate is 9%. The floating-rate note would require

$$0.25 \times 0.09 \times \$10{,}000{,}000 = \$225{,}000$$

of interest to be paid three months later. With a three-month LIBOR rate of 8% the interest payment would be

$$0.25 \times 0.08 \times \$10{,}000{,}000 = \$200{,}000$$

The cap, therefore, provides a payoff of $25,000 (= $225,000 – $200,000).[3] Note that the payoff does not occur on the reset date when the 9% is observed. It occurs three months later. This reflects the usual time lag between an interest rate being observed and the corresponding payment being required.

If a corporation obtains a floating-rate loan where the rate of interest is linked to LIBOR, a cap can be used to limit the interest paid. For example, if the floating rate on a loan is LIBOR plus 30 basis points and the loan lasts for five years, the cap we have just considered would ensure that the rate paid is never higher than 8.30%. At each reset date during the life of the cap we observe LIBOR. If LIBOR is less than 8%, there is no payoff from the cap three months later. If LIBOR is greater than 8%, the payoff is one-quarter of the excess applied to the principal of $10 million. Note that caps are usually defined so that the initial LIBOR rate, even if it is greater than the cap rate, does not lead to a payoff on the first reset date. In our example the cap lasts for five years. There are, therefore, a total of 19 reset dates (at times 0.25, 0.5, 0.75, ..., 4.75 years) and 19 potential payoffs from the caps (at times 0.50, 0.75, 1.00, ..., 5.00 years). This example is summarized in Table 18.1.

The Cap as a Portfolio of Interest Rate Options

Consider a cap with a principal of L, and a cap rate of R_X. Suppose that the reset dates are $t_1, t_2, \ldots, t_n$ and the corresponding payment dates are $t_2, t_3, \ldots, t_{n+1}$. Define R_k as the interest rate for the period between time t_k and t_{k+1} observed at time t_k $(1 \leqslant k \leqslant n)$.

[3] This calculation assumes exactly one-quarter of a year between reset dates. In practice the calculation takes account of the exact number of days between reset dates using a specified day count convention.

The cap leads to a payoff at time t_{k+1} of

$$L\delta_k \max(R_k - R_X, 0) \tag{18.7}$$

where $\delta_k = t_{k+1} - t_k$.[4]

Equation (18.7) is a call option on the LIBOR rate observed at time t_k with the payoff occurring at time t_{k+1}. The cap is a portfolio of n such call options. These call options are known as *caplets*.

Floors and Collars

Interest rate floors and interest rate collars (sometimes called floor–ceiling agreements) are defined analogously to caps. A *floor* provides a payoff when the interest rate on the underlying floating-rate note falls below a certain rate. With the notation already introduced, a floor provides a payoff at time t_{k+1} ($k = 1, 2, \dots, n$) of

$$L\delta_k \max(R_X - R_k, 0)$$

Analogously to an interest rate cap, an interest rate floor is a portfolio of put options on interest rates. Each of the individual options comprising a floor is known as a *floorlet*. A *collar* is an instrument designed to guarantee that the interest rate on the underlying floating-rate note always lies between two levels. A collar is a combination of a long position in a cap and a short position in a floor. It is usually constructed so that the price of the cap is initially equal to the price of the floor. The cost of entering into the collar is then zero.

There is a put–call parity relationship between the prices of caps and floors. This is

$$\text{cap price} = \text{floor price} + \text{value of swap}$$

In this relationship, the cap and floor have the same strike price, R_X. The swap is an agreement to receive floating and pay a fixed rate of R_X with no exchange of payments on the first reset date.[5] All three instruments have the same life and the same frequency of payments. This result can be seen to be true by noting that a long position in the cap combined with a short position in the floor provides the same cash flows as the swap.

Valuation of Caps and Floors

As shown in equation (18.7), the caplet corresponding to the rate observed at time t_k provides a payoff at time t_{k+1} of

$$L\delta_k \max(R_k - R_X, 0)$$

If the rate R_k is assumed to be lognormal with volatility σ_k, equation (18.3) gives the

[4] In this equation both R_k and R_X are expressed with a compounding frequency equal to the frequency of resets. Also, it is assumed that they are measured on an actual/actual day count basis. In the United States LIBOR is quoted on an actual/360 basis. For the purposes of equation (18.7) and other equations in this chapter, we assume that quotes have been multiplied by 365/360 or 366/360 to convert them to an actual/actual basis.

[5] Note that swaps are usually structured so that the rate at time zero determines an exchange of payments at the first reset date. As indicated earlier, caps and floors are usually structured so that there is no payoff at the first reset date. This difference explains why we have to exclude the first exchange of payments on the swap.

value of this caplet as

$$L\delta_k e^{-r_{k+1}t_{k+1}}[F_k N(d_1) - R_X N(d_2)] \tag{18.8}$$

where r_{k+1} is the continuously compounded rate for a maturity t_{k+1}.

$$d_1 = \frac{\ln(F_k/R_X) + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}}$$

$$d_2 = \frac{\ln(F_k/R_X) - \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}} = d_1 - \sigma_k\sqrt{t_k}$$

and F_k is the forward rate for the period between time t_k and t_{k+1}. The value of the corresponding floorlet is, from equation (18.4),

$$L\delta_k e^{-r_{k+1}t_{k+1}}[R_X N(-d_2) - F_k N(-d_1)] \tag{18.9}$$

Note that R_X and F_k are expressed with a compounding frequency equal to the frequency of resets in these equations.

Example

Consider a contract that caps the interest rate on a \$10,000 loan at 8% per annum (with quarterly compounding) for three months starting in one year. This is a caplet and could be one element of a cap. Suppose that the zero curve is flat at 7% per annum with quarterly compounding and the one-year volatility for the three-month rate underlying the caplet is 20% per annum. The continuously compounded zero rate for all maturities is 6.9394%. In equation (18.8), $F_k = 0.07$, $\delta_k = 0.25$, $L = 10,000$, $R_X = 0.08$, $r_{k+1} = 0.069394$, and $\sigma_k = 0.20$, $t_k = 1.0$, $t_{k+1} = 1.25$. Also

$$d_1 = \frac{\ln(0.07/0.08) + 0.2^2 \times 1/2}{0.2 \times 1} = -0.5677$$

$$d_2 = d_1 - 0.20 = -0.7677$$

so that the caplet price is

$$0.25 \times 10{,}000 \times e^{-0.069394 \times 1.25}[0.07N(-0.5677) - 0.08N(-0.7677)] = \$5.162$$

(Note that DerivaGem gives \$5.146 for the price of this caplet. This is because it assumes 365 days per year and rounds times to the nearest whole number of days.)

Each caplet of a cap must be valued separately using equation (18.8). One approach is to use a different volatility for each caplet. The volatilities are then referred to as *spot volatilities*.[6] An alternative approach is to use the same volatility for all the caplets comprising any particular cap, but to vary this volatility according to the life of the cap. The volatilities used are then referred to as *flat volatilities*. The volatilities quoted in the market are usually flat volatilities. However, many traders like to work with spot volatilities because this allows them to identify underpriced and overpriced caplets and floorlets. The options on Eurodollar futures that trade on the Chicago Mercantile Exchange are similar to caplets. The implied spot volatilities for caplets on three-month

[6] The term *forward volatilities* is sometimes also used to describe these volatilities.

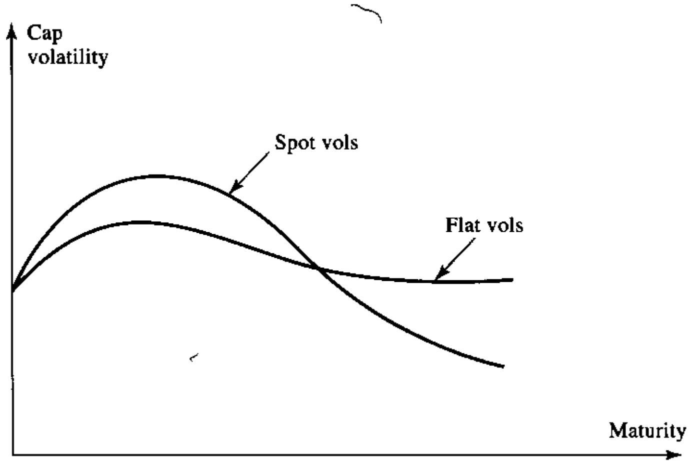

Figure 18.4 The volatility hump

LIBOR are frequently compared with those calculated from the prices of Eurodollar futures options.

Figure 18.4 shows a typical pattern for spot volatilities and flat volatilities as a function of maturity. (In the case of a spot volatility, the maturity is the maturity of a caplet; in the case of a flat volatility, it is the maturity of a cap.) The flat volatilities are akin to cumulative averages of the spot volatilities and, therefore, exhibit less variability. As indicated by Figure 18.4, we usually observe a "hump" in the volatilities at about the two- to three-year point. This hump is observed both when the volatilities are implied from option prices and when they are calculated from historical data. There is no general agreement on the reason for the existence of the hump. One possible explanation is as follows. Rates at the short end of the zero curve are controlled by central banks. By contrast, two- and three-year interest rates are determined to a large extent by the activities of traders. These traders may be overreacting to the changes they observe in the short rate and causing the volatility of these rates to be higher than the volatility of short rates. For maturities beyond two to three years the mean reversion of interest rates, which will be discussed later in this chapter, causes volatilities to decline.

Brokers provide tables of flat implied volatilities for caps and floors. The instruments underlying the quotes are usually at the money. This means that the cap/floor rate equals the swap rate for a swap that has the same payment dates as the cap. Table 18.2 shows typical broker quotes for the U.S. dollar market. The tenor of the cap is

Table 18.2 Typical broker volatility quotes for U.S. dollar caps and floors (percent per annum)

Life	Cap bid	Cap offer	Floor bid	Floor offer
1 year	18.00	20.00	18.00	20.00
2 year	23.25	24.25	23.75	24.75
3 year	24.00	25.00	24.50	25.50
4 year	23.75	24.75	24.25	25.25
5 year	23.50	24.50	24.00	25.00
7 year	21.75	22.75	22.00	23.00
10 year	20.00	21.00	20.25	21.25

three months, and the cap life varies from 1 year to 10 years. The volatilities are flat volatilities rather than spot volatilities. The data exhibits the type of "hump" shown in Figure 18.4.

Use of DerivaGem

The software DerivaGem accompanying this book can be used to price interest rate caps and floors using Black's model. In the Cap_and_Swap_Option worksheet select Cap/Floor as the Underlying Type and Black-European as the Pricing Model. The zero curve is input using continuously compounded rates. The inputs include the start date and the end date of the period covered by the cap, the flat volatility, and the cap settlement frequency (that is, the tenor). The software calculates the payment dates by working back from the end of the period covered by the cap to the beginning. The initial caplet/floorlet is assumed to cover a period of length between 0.5 and 1.5 times a regular period. Suppose, for example, that the period covered by the cap is 1.2 years to 2.8 years and the settlement frequency is quarterly. There are six caplets covering the periods 2.55 years to 2.80 years, 2.30 years to 2.55 years, 2.05 years to 2.30 years, 1.80 years to 2.05 years, 1.55 years to 1.80 years, and 1.20 years to 1.55 years.

18.6 EUROPEAN SWAP OPTIONS

Swap options, or *swaptions*, are options on interest rate swaps and are an increasingly popular type of interest rate option. They give the holder the right to enter into a specified interest rate swap at a certain time in the future. (The holder does not have to exercise this right.) Many large financial institutions that offer interest rate swap contracts to their corporate clients are also prepared to sell them swaptions or buy swaptions from them.

To give an example of how a swaption might be used, consider a company that knows that in six months it will enter into a five-year floating-rate loan agreement and that it will wish to swap the floating-interest payments for fixed-interest payments to convert the loan into a fixed-rate loan. (See Chapter 6 for a discussion of how swaps can be used in this way.) At a cost the company could enter into a swaption, giving it the right to receive six-month LIBOR and pay a certain fixed rate of interest (say, 6% per annum) for a five-year period starting in six months. If the fixed rate on a regular five-year swap in six months turns out to be less than 6% per annum, the company will choose not to exercise the swaption and will enter into a swap agreement in the usual way. However, if the fixed rate turns out to be greater than 6% per annum, the company will choose to exercise the swaption and will obtain a swap at more favorable terms than those available in the market. This example is summarized in Table 18.3.

When used in the way just described, swaptions provide companies that are planning future borrowings with protection against interest rate increases. Swaptions are an alternative to forward swaps (sometimes called *deferred swaps*). Forward swaps involve no up-front cost, but have the disadvantage that they obligate the company to enter into a swap agreement. With a swaption, the company is able to benefit from favorable interest rate movements while acquiring protection from unfavorable interest rate movements. The difference between a swaption and a forward swap is analogous to

Table 18.3 Use of a swaption

From the Trader's Desk

A company knows that it will be entering into a five-year floating-rate loan agreement in six months and plans to swap the floating-interest payments for fixed-interest payments. It would like to ensure that the fixed rate exchanged for LIBOR in the swap is no more than 6%.

The Strategy

The company buys a swaption. The swaption gives it the right (but not the obligation) to enter into a swap where LIBOR is received and 6% is paid for a five-year period starting in six months. If the fixed rate on a regular five-year swap in six-months' time turns out to be greater than 6% per annum, the company will exercise the swaption; under other circumstances, it will choose to negotiate a swap reflecting market rates of interest.

the difference between an option on foreign exchange and a forward contract on foreign exchange.

Relation to Bond Options

Recall from Chapter 6 that an interest rate swap can be regarded as an agreement to exchange a fixed-rate bond for a floating-rate bond. At the start of a swap, the value of the floating-rate bond always equals the principal amount of the swap. A swaption can therefore be regarded as an option to exchange a fixed-rate bond for the principal amount of the swap. If a swaption gives the holder the right to pay fixed and receive floating, it is a put option on the fixed-rate bond with strike price equal to the principal. If a swaption gives the holder the right to pay floating and receive fixed, it is a call option on the fixed-rate bond with a strike price equal to the principal.

Valuation of European Swaptions

As explained in Chapter 6, the *swap rate* for a particular maturity at a particular time is the fixed rate that would be exchanged for LIBOR in a newly issued swap with that maturity. The model usually used to value a European option on a swap assumes that the relevant swap rate at the maturity of the option is lognormal. Consider a swaption where we have the right to pay a rate R_X and receive LIBOR on a swap that will last n years starting in T years. We suppose that there are m payments per year under the swap and that the principal is L.

Suppose that the swap rate for an n-year swap at the maturity of the swap option is R. (Both R and R_X are expressed with a compounding frequency of m times per year.) By comparing the cash flows on a swap where the fixed rate is R to the cash flows on a swap where the fixed rate is R_X, we see that the payoff from the swaption consists of a series of cash flows equal to

$$\frac{L}{m}\max(R - R_X,\ 0)$$

The cash flows are received m times per year for the n years of the life of the swap. Suppose that the payment dates are $t_1, t_2, \ldots, t_{mn}$, measured in years from today. (It is approximately true that $t_i = T + i/m$.) Each cash flow is the payoff from a call option on R with strike price R_X.

Using equation (18.3), the value of the cash flow received at time t_i is

$$\frac{L}{m} e^{-r_i t_i}[F_0 N(d_1) - R_X N(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/R_X) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/R_X) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

F_0 is the forward swap rate, and r_i is the continuously compounded zero-coupon interest rate for a maturity of t_i.

The total value of the swaption is

$$\sum_{i=1}^{mn} \frac{L}{m} e^{-r_i t_i}[F_0 N(d_1) - R_X N(d_2)]$$

If we define A as the value of a contract that pays $1/m$ at times t_i $(1 \leqslant i \leqslant mn)$ so that

$$A = \frac{1}{m}\sum_{i=1}^{mn} e^{-r_i t_i}$$

the value of the swaption becomes

$$LA[F_0 N(d_1) - R_X N(d_2)] \tag{18.10}$$

If the swaption gives the holder the right to receive a fixed rate of R_X instead of paying it, the payoff from the swaption is

$$\frac{L}{m}\max(R_X - R,\ 0)$$

This is a put option on R. As before, the payoffs are received at times t_i $(1 \leqslant i \leqslant mn)$. Equation (18.4) gives the value of the swaption as

$$LA[R_X N(-d_2) - F_0 N(-d_1)] \tag{18.11}$$

The DerivaGem software provides an implementation of equations (18.10) and (18.11). In the Cap_and_Swap_Options worksheet select Swap Option as the Underlying Type and Black-European as the Pricing Model.

Example

Suppose that the LIBOR yield curve is flat at 6% per annum with continuous compounding. Consider a swaption that gives the holder the right to pay 6.2% in a three-year swap starting in five years. The volatility for the swap rate is 20%. Payments are made semiannually and the principal is \$100. In this case

$$A = \tfrac{1}{2}(e^{-0.06\times5.5} + e^{-0.06\times6} + e^{-0.06\times6.5} + e^{-0.06\times7} + e^{-0.06\times7.5} + e^{-0.06\times8}) = 2.0035$$

A rate of 6% per annum with continuous compounding translates into 6.09%

Table 18.4 Typical broker quotes for U.S. European swap options (mid-market volatilities % per annum)

	Swap length						
Expiration	1 Year	2 Year	3 Year	4 Year	5 Year	7 Year	10 Year
1 month	17.75	17.75	17.75	17.50	17.00	17.00	16.00
3 month	19.50	19.00	19.00	18.00	17.50	17.00	16.00
6 month	20.00	20.00	19.25	18.50	18.75	17.75	16.75
1 year	22.50	21.75	20.50	20.00	19.50	18.25	16.75
2 year	22.00	22.00	20.75	19.50	19.75	18.25	16.75
3 year	21.50	21.00	20.00	19.25	19.00	17.75	16.50
4 year	20.75	20.25	19.25	18.50	18.25	17.50	16.00
5 year	20.00	19.50	18.50	17.75	17.50	17.00	15.50

with semiannual compounding. It follows that in this example $F_0 = 0.0609$, $R_X = 0.062$, $T = 5$, and $\sigma = 0.2$, so that

$$d_1 = \frac{\ln(0.0609/0.062) + 0.2^2 \times 5/2}{0.2\sqrt{5}} = 0.1836$$

$$d_2 = d_1 - 0.2\sqrt{5} = -0.2636$$

From equation (18.10) the value of the swaption is

$$100 \times 2.0035[0.0609 \times N(0.1836) - 0.062 \times N(-0.2636)] = 2.07$$

or \$2.07. (This is in agreement with the price given by DerivaGem.)

Brokers provide tables of implied volatilities for European swap options. The instruments underlying the quotes are usually at the money. This means that the strike swap rate equals the forward swap rate. Table 18.4 shows typical broker quotes provided for the U.S. dollar market. The tenor of the underlying swaps (that is, the frequency of resets on the floating rate) is six months. The life of the option is shown on the vertical scale. This varies from one month to five years. The life of the underlying swap at the maturity of the option is shown on the horizontal scale. This varies from 1 year to 10 years. The volatilities in the extreme left column of the table correspond to instruments that are similar to caps. They exhibit the hump discussed earlier. As we move to the columns corresponding to options on longer-lived swaps, the hump persists, but it becomes less pronounced.

18.7 TERM STRUCTURE MODELS

The European bond option pricing model that we have presented assumes that a bond price at some future time is lognormally distributed; the cap pricing model assumes that an interest rate at some future time is lognormally distributed; the European swap option pricing model assumes that a swap rate at some future time is lognormally distributed. These assumptions are not consistent with each other. This makes it difficult for traders to compare the way the market prices different types of instruments.

A related disadvantage of the models is that they cannot easily be extended to value instruments other than those for which they were designed. For example, Black's model

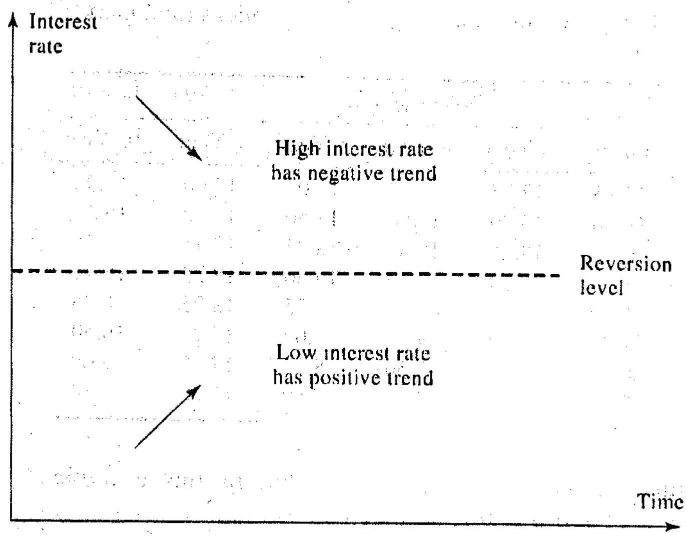

Figure 18.5 Mean reversion

for valuing a European swap option cannot easily be extended to value American swap options. A more sophisticated approach to valuing interest rate derivative securities involves constructing a *term structure model*. This is a model that describes the probabilistic behavior of the term structure of interest rates. Term structure models are more complicated than the models used to describe the movements of a stock price or currency. This is because they are concerned with movements in the whole zero-coupon yield curve—not with changes to a single variable. As time passes, all interest rates do not necessarily change by the same amount so that the shape of the yield curve is liable to change.

It is beyond the scope of this book to provide a complete description of how yield curve models are constructed. But it is worth noting one property of an interest rate that distinguishes it from a stock price or an exchange rate—or indeed the price of any investment asset. A short-term interest rate (say, the three-month rate) appears to exhibit a property known as *mean reversion*. It tends to be pulled back to some long-run average level. When the short-term interest rate is very high, it tends to move down; when it is very low, it tends to move up. For example, if the three-month interest rate in the United States reaches 15%, the next movement is more likely to be down than up; if it reaches 2%, the next movement is more likely to be up than down. This is illustrated in Figure 18.5.

If a stock price exhibited mean reversion, there would be an obvious trading strategy for us to follow: buy the stock when its price is at a historic low; sell the stock when its price is at a historic high. Mean reverting three-month interest rates do not provide us with a similar trading strategy. This is because an interest rate is not the price of a security that can be traded. There is no traded instrument whose price is always equal the three-month rate.

18.8 SUMMARY

Interest rate options arise in practice in many different ways. For example, options on Treasury bond futures, Treasury note futures, and Eurodollar futures are actively traded by exchanges. Many traded bonds include features that are options. The loans and deposit instruments offered by financial institutions often contain embedded options.

Three popular over-the-counter instruments are bond options, interest caps and floors, and swap options. A bond option is an option to buy or sell a particular bond. An interest rate cap (floor) provides a payoff when a floating rate of interest rises above (falls below) the strike rate. A swap option is an option to enter into a swap, where a specified fixed rate will be exchanged for floating, at a particular time in the future. Black's model is the model used by the market for valuing these instruments. In the case of bond options, the probability distribution of the underlying bond is assumed to be lognormal. In the case of caps and floors, the underlying interest rates are assumed to be lognormal. In the case of swap options, the underlying swap rate is assumed to be lognormal.

Suggestions for Further Reading

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Quiz (Answers at End of Book)

18.1. A company caps three-month LIBOR at 10% per annum. The principal amount is $20 million. On a reset date three-month LIBOR is 12% per annum. What payment would this lead to under the cap? When would the payment be made?

18.2. Explain the features of (a) callable and (b) puttable bonds.

18.3. Explain why a swaption can be regarded as a type of bond option.

18.4. Use Black's model to value a 1-year European put option on a 10-year bond. Assume that the current value of the bond is \$125, the strike price is \$110, the 1-year interest rate is 10% per annum, the bond's price volatility is 8% per annum, and the present value of the coupons that will be paid during the life of the option is \$10.

18.5. Suppose that the LIBOR yield curve is flat at 8% with annual compounding. A swaption gives the holder the right to receive 7.6% in a five-year swap starting in four years. Payments are made annually. The volatility for the swap rate is 25% per annum and the principal is \$1 million. Use Black's model to price the swaption.

18.6. Calculate the price of an option that caps the 3-month rate starting in 18-months' time at 13% (quoted with quarterly compounding) on a principal amount of \$1,000. The forward interest rate for the period in question is 12% per annum (quoted with quarterly compounding), the 21-month risk-free interest rate (continuously compounded) is 11.5% per annum, and the volatility of the forward rate is 12% per annum.

18.7. What are the advantages of yield curve models over the Black's model for valuing interest rate derivatives?

Questions and Problems (Answers in Solutions Manual)

18.8. A bank uses Black's model to price European bond options. Suppose that an implied price volatility for a 5-year option on a bond maturing in 10 years is used to price a 9-year option on the bond. Would you expect the resultant price to be too high or too low? Why?

18.9. Consider a four-year European call option on a bond that will mature in five years. The five-year bond price is \$105, the price of a four-year bond with the same coupon as the five-year bond is \$102, the strike price of the option is \$100, the four-year risk-free interest rate is 10% per annum (continuously compounded), and the volatility of the forward price of the bond underlying the option is 2% per annum. What is the present value of the principal in the four-year bond? What is the present value of the coupons in the four-year bond? What is the forward price of the bond underlying the option? What is the value of the option?

18.10. If the yield volatility for a five-year put option on a bond maturing in 10 years time is specified as 22%, how should the option be valued? Assume that, based on today's interest rates the modified duration of the bond at the maturity of the option will be 4.2 years and the forward yield on the bond is 7%.

18.11. A corporation knows that in three months it will have \$5 million to invest for 90 days at LIBOR minus 50 basis points and wishes to ensure that the rate obtained will be at least 6.5%. What position in exchange-traded interest rate options should the bank take?

18.12. Explain carefully how you would use (a) spot volatilities and (b) flat volatilities to value a five-year cap.

18.13. What other instrument is the same as a five-year zero-cost collar in which the strike price of the cap equals the strike price of the floor? What does the common strike price equal?

18.14. Suppose that the 1-year, 2-year, 3-year, 4-year and 5-year zero rates are 6%, 6.4%, 6.7%, 6.9%, and 7%. The price of a 5-year semiannual cap with a principal of \$100 at a cap rate of 8% is \$3. Use DerivaGem to determine

a. The 5-year flat volatility for caps and floors

b. The floor rate in a zero-cost 5-year collar when the cap rate is 8%

18.15. Show that $V_1 + f = V_2$ where V_1 is the value of a swap option to pay a fixed rate of R_X and receive LIBOR between times T_1 and T_2, f is the value of a forward swap to receive a fixed rate of R_X and pay LIBOR between times T_1 and T_2, and V_2 is the value of a swap option to receive a fixed rate of R_X between times T_1 and T_2. Deduce that $V_1 = V_2$ when R_X equals the current forward swap rate.

18.16. Explain why there is an arbitrage opportunity if the implied Black (flat) volatility for a cap is different from that for a floor. Do the broker quotes in Table 18.2 present an arbitrage opportunity?

18.17. Suppose that zero rates are as in Problem 18.14. Use DerivaGem to determine the value of an option to pay a fixed rate of 6% and receive LIBOR on a five-year swap starting in one year. Assume that the principal is \$100 million, payments are exchanged semi-annually, and the swap rate volatility is 21%.

Assignment Questions

18.18. Consider an eight-month European put option on a Treasury bond that currently has 14.25 years to maturity. The bond principal is \$1,000. The current cash bond price is \$910, the exercise price is \$900, and the volatility of the forward bond price is 10% per annum. A coupon of \$35 will be paid by the bond in three months. The risk-free interest rate is 8% for all maturities up to one year. Use Black's model to determine the price of the option. Consider both the case where the strike price corresponds to the cash price of the bond and the case where it corresponds to the quoted price.

18.19. Calculate the price of a cap on the three-month LIBOR rate in nine months' time for a principal amount of \$1,000. Use Black's model and the following information:
Quoted nine-month Eurodollar futures price = 92
Interest rate volatility implied by a nine-month Eurodollar option = 15% per annum
Current 12-month interest rate with continuous compounding = 7.5% per annum
Cap rate = 8% per annum

18.20. Suppose that the LIBOR yield curve is flat at 8% with annual compounding. A swaption gives the holder the right to receive 7.6% in a five-year swap starting in four years. Payments are made annually. The volatility for the swap rate is 25% per annum and the principal is \$1 million. Use Black's model to price the swaption. Compare your answer to that given by DerivaGem.

18.21. Use the DerivaGem software to value a five-year collar that guarantees that the maximum and minimum interest rates on a LIBOR-based loan (with quarterly resets) are 5% and 7%, respectively. The LIBOR zero curve (continuously compounded) is currently flat at 6%. Use a flat volatility of 20%. Assume that the principal is \$100.

18.22. Use the DerivaGem software to value a European swap option that gives you the right in two years to enter into a 5-year swap in which you pay a fixed rate of 6% and receive floating. Cash flows are exchanged semiannually on the swap. The 1-year, 2-year, 5-year, and 10-year zero-coupon interest rates (continuously compounded) are 5%, 6%, 6.5%, and 7%, respectively. Assume a principal of \$100 and a volatility of 15% per annum. Give an example of how the swap option might be used by a corporation. What bond option is equivalent to the swap option?