CHAPTER 3

Determination of Forward and Futures Prices

In this chapter we examine how forward prices and futures prices are related to the spot price of the underlying asset. Forward contracts are easier to analyze than futures contracts because there is no daily settlement—only a single payment at maturity. Consequently, most of the analysis in the first part of the chapter is directed toward determining forward prices rather than futures prices. Luckily it can be shown that the forward price and futures price of an asset are usually very close when the maturities of the two contracts are the same. In the second part of the chapter we use this result to examine the properties of futures prices for contracts on stock indices, foreign exchange, and other assets.

3.1 INVESTMENT ASSETS vs. CONSUMPTION ASSETS

When considering forward and futures contracts, it is important to distinguish between investment assets and consumption assets. An *investment asset* is an asset that is held for investment purposes by significant numbers of investors. Stocks and bonds are clearly investment assets. Gold and silver are also examples of investment assets. Note that investment assets do not have to be held exclusively for investment. Silver, for example, has a number of industrial uses. However, they do have to satisfy the requirement that they are held by significant numbers of investors solely for investment. A *consumption asset* is an asset that is held primarily for consumption. It is not usually held for investment. Examples of consumption assets are commodities such as copper, oil, and pork bellies.

As we will see later in this chapter, we can use arbitrage arguments to determine the forward and futures prices of an investment asset from its spot price and other observable market variables. We cannot do this for the forward and futures prices of consumption assets.

3.2 SHORT SELLING

Some of the arbitrage strategies presented in this chapter involve *short selling*. This trade, usually simply referred to as "shorting," involves selling an asset that is not

Table 3.1 Example of a short sale

From the Trader's Desk

An investor shorts 500 IBM shares in April when the price is \$120 and buys them back (to close out the position) in July when the price is \$100. A dividend of \$1 per share is paid in May.

The Profit

The investor receives 500 × \$120 in April and must pay 500 × \$1 in May. The cost of closing out the position is 500 × \$100. The net gain (ignoring the time value of money) is, therefore,

$$(500 \times \$120) - (500 \times \$1) - (500 \times \$100) = \$9{,}500$$

owned. It is something that is possible for some—but not all—investment assets. We will illustrate how it works by considering a short sale of shares of a stock.

Suppose an investor instructs a broker to short 500 IBM shares. The broker will carry out the instructions by borrowing the shares from another client and selling them in the market in the usual way. The investor can maintain the short position for as long as desired, provided there are always shares for the broker to borrow. At some stage, however, the investor will close out the position by purchasing 500 IBM shares. These are then replaced in the account of the client from which the shares were borrowed. The investor takes a profit if the stock price has declined and a loss if it has risen. If, at any time while the contract is open, the broker runs out of shares to borrow, the investor is *short-squeezed* and is forced to close out the position immediately even if not ready to do so.

An investor with a short position must pay to the broker any income, such as dividends or interest, that would normally be received on the securities that have been shorted. The broker will transfer this to the account of the client from whom the securities have been borrowed. Consider the position of an investor who shorts 500 IBM shares in April when the price per share is \$120 and closes out the position by buying them back in July when the price per share is \$100. Suppose that a dividend of \$1 per share is paid in May. The investor receives 500 × \$120 = \$60,000 in April when the short position is initiated. The dividend leads to a payment by the investor of 500 × \$1 = \$500 in May. The investor also pays 500 × \$100 = \$50,000 when the position is closed out in July. The net gain is, therefore,

$$\$60{,}000 - \$500 - \$50{,}000 = \$9{,}500$$

This example is summarized in Table 3.1.

The investor is required to maintain a *margin account* with the broker. The margin account consists of cash or marketable securities deposited by the investor with the broker to guarantee that the investor will not walk away from the short position if the share price increases. It is similar to the margin account discussed in Chapter 2 for futures contracts. An initial margin is required and if there are adverse movements (i.e., increases) in the price of the asset that is being shorted, additional margin may be required. The margin account does not represent a cost to the investor. This is because interest is usually paid on the balance in margin accounts and, if the interest rate offered is unacceptable, marketable securities such as Treasury bills can be used to meet margin requirements. The proceeds of the sale of the asset belong to the investor and normally form part of the initial margin.

Regulators in the United States currently allow a stock to be shorted only on an *uptick*—that is, when the most recent movement in the price of the stock was an increase. An exception is made when traders are shorting a basket of stocks replicating a stock index.

3.3 MEASURING INTEREST RATES

Before getting into the details of how forward and futures prices are determined, it is important to talk about how interest rates are measured. A statement by a bank that the interest rate on one-year deposits is 10% per annum may sound straightforward and unambiguous. In fact, its precise meaning depends on the way the interest rate is measured.

If the interest rate is quoted with annual compounding, the bank's statement that the interest rate is 10% means that \$100 grows to

$$\$100 \times 1.1 = \$110$$

at the end of one year. When the interest rate is expressed with semiannual compounding, it means that we earn 5% every six months, with the interest being reinvested. In this case \$100 grows to

$$\$100 \times 1.05 \times 1.05 = \$110.25$$

at the end of one year. When the interest rate is expressed with quarterly compounding, the bank's statement means that we earn 2.5% every three months, with the interest being reinvested. The \$100 then grows to

$$\$100 \times 1.025^4 = \$110.38$$

at the end of one year. Table 3.2 shows the effect of increasing the compounding frequency further.

The compounding frequency defines the units in which an interest rate is measured. A rate expressed with one compounding frequency can be converted into an equivalent rate with a different compounding frequency. For example, from Table 3.2 we see that 10.25% with annual compounding is equivalent to 10% with semiannual compounding.

Table 3.2 Effect of the compounding frequency on the value of \$100 at the end of one year when the interest rate is 10% per annum

Compounding frequency	Value of \$100 at end of year (\$)
Annually ($m = 1$)	110.00
Semiannually ($m = 2$)	110.25
Quarterly ($m = 4$)	110.38
Monthly ($m = 12$)	110.47
Weekly ($m = 52$)	110.51
Daily ($m = 365$)	110.52

We can think of the difference between one compounding frequency and another to be analogous to the difference between kilometers and miles.

To generalize our results, suppose that an amount A is invested for n years at an interest rate of R per annum. If the rate is compounded once per annum, the terminal value of the investment is

$$A(1+R)^n$$

If the rate is compounded m times per annum, the terminal value of the investment is

$$A\left(1+\frac{R}{m}\right)^{mn} \tag{3.1}$$

Continuous Compounding

The limit as m tends to infinity is known as *continuous compounding*. With continuous compounding, it can be shown that an amount A invested for n years at rate R grows to

$$Ae^{Rn} \tag{3.2}$$

where $e = 2.71828$. The function e^x is built into most calculators, so the computation of the expression in equation (3.2) presents no problems. In the example in Table 3.2, $A = 100$, $n = 1$, and $R = 0.1$, so that the value to which A grows with continuous compounding is

$$100e^{0.1} = \$110.52$$

This is (to two decimal places) the same as the value with daily compounding. For most practical purposes, continuous compounding can be thought of as being equivalent to daily compounding. Compounding a sum of money at a continuously compounded rate R for n years involves multiplying it by e^{Rn}. Discounting it at a continuously compounded rate R for n years involves multiplying by e^{-Rn}.

In this book interest rates will be measured with continuous compounding except where otherwise stated. Readers used to working with interest rates that are measured with annual, semiannual, or some other compounding frequency may find this a little strange at first. However, continuously compounded interest rates are used to such a great extent in pricing derivatives that it makes sense to get used to working with them now.

Suppose that R_c is a rate of interest with continuous compounding and R_m is the equivalent rate with compounding m times per annum. From the results in equations (3.1) and (3.2), we must have

$$Ae^{R_c n} = A\left(1+\frac{R_m}{m}\right)^{mn}$$

or

$$e^{R_c} = \left(1+\frac{R_m}{m}\right)^{m}$$

This means that

$$R_c = m\ln\left(1+\frac{R_m}{m}\right) \tag{3.3}$$

and

$$R_m = m(e^{R_c/m} - 1) \tag{3.4}$$

These equations can be used to convert a rate with a compounding frequency of m times per annum to a continuously compounded rate and vice versa. The function ln is the natural logarithm function and is built into most calculators. It is defined so that if $y = \ln x$, then $x = e^y$.

Examples

1. Consider an interest rate that is quoted as 10% per annum with semiannual compounding. From equation (3.3) with $m = 2$ and $R_m = 0.1$, the equivalent rate with continuous compounding is

$$2\ln\left(1 + \frac{0.1}{2}\right) = 0.09758$$

or 9.758% per annum.

2. Suppose that a lender quotes the interest rate on loans as 8% per annum with continuous compounding, and that interest is actually paid quarterly. From equation (3.4) with $m = 4$ and $R_c = 0.08$, the equivalent rate with quarterly compounding is

$$4(e^{0.08/4} - 1) = 0.0808$$

or 8.08% per annum. This means that on a \$1,000 loan, interest payments of \$20.20 would be required each quarter.

3.4 ASSUMPTIONS AND NOTATION

In this chapter we will assume that the following are all true for some market participants:

1. The market participants are subject to no transactions costs when they trade.
2. The market participants are subject to the same tax rate on all net trading profits.
3. The market participants can borrow money at the same risk-free rate of interest as they can lend money.
4. The market participants take advantage of arbitrage opportunities as they occur.

Note that we do not require these assumptions to be true for all market participants. All that we require is that they be true—or at least approximately true—for a few key market participants such as large investment banks. This is not unreasonable. It is the trading activities of these key market participants and their eagerness to take advantage of arbitrage opportunities as they occur that determine the relationship between forward and spot prices.

The following notation will be used throughout this chapter:

T: Time until delivery date in a forward or futures contract (in years)

S_0: Price of the asset underlying the forward or futures contract today

F_0: Forward or futures price today

r: Risk-free rate of interest per annum, expressed with continuous compounding, for an investment maturing at the delivery date (i.e., in T years)

The risk-free rate, r, is in theory the rate at which money is borrowed or lent when there is no credit risk so that the money is certain to be repaid. It is often thought of as the Treasury rate; that is, the rate at which a national government borrows in its own

Table 3.3 Arbitrage opportunity when forward price of a non-dividend-paying stock is too high

From the Trader's Desk

The forward price of a stock for a contract with delivery date in three months is \$43. The three-month risk-free interest rate is 5% per annum, and the current stock price is \$40. No dividends are expected.

Opportunity

The forward price is too high relative to the stock price. An arbitrageur can

1. Borrow \$40 to buy one share spot.
2. Enter into a forward contract to sell one share in three months.

At the end of three months, the arbitrageur delivers the share and receives \$43. The sum of money required to pay off the loan is $40e^{0.05\times 3/12} = \$40.50$. The arbitrageur therefore makes a profit at the end of the three-month period of

$$\$43 - \$40.50 = \$2.50$$

currency. In practice, large financial institutions usually set r equal to the London Interbank Offer Rate (LIBOR) instead of the Treasury rate in the formulas in this chapter—and in those in the rest of the book. LIBOR will be discussed in Chapter 5. It is the rate paid by one bank when it borrows from another bank.

3.5 FORWARD PRICE FOR AN INVESTMENT ASSET

The easiest forward contract to value is one written on an investment asset that provides the holder with no income. Non-dividend-paying stocks and zero-coupon bonds are examples of such investment assets.[1]

Illustration

Consider a long forward contract to purchase a non-dividend-paying stock in three months. Assume the current stock price is \$40 and the three-month risk-free interest rate is 5% per annum. We consider strategies open to an arbitrageur in two extreme situations.

Suppose first that the forward price is relatively high at \$43. An arbitrageur can borrow \$40 at the risk-free interest rate of 5% per annum, buy one share, and short a forward contract to sell one share in three months. At the end of the three months, the arbitrageur delivers the share and receives \$43. The sum of money required to pay off the loan is

$$40e^{0.05\times 3/12} = \$40.50$$

By following this strategy, the arbitrageur locks in a profit of \$43.00 – \$40.50 = \$2.50 at the end of the three-month period. The strategy is summarized in Table 3.3.

Suppose next that the forward price is relatively low at \$39. An arbitrageur can short one share, invest the proceeds of the short sale at 5% per annum for three months, and

[1] Some of the contracts mentioned in the first half of this chapter (e.g., forward contracts on individual stocks) do not normally arise in practice. However, they form useful examples for developing our ideas.

Table 3.4 Arbitrage opportunity when forward price of a non-dividend-paying stock is too low

From the Trader's Desk

The forward price of a stock for a contract with a delivery date in three months is \$39. The three-month risk-free interest rate is 5% per annum and the current stock price is \$40. No dividends are expected.

Opportunity

The forward price is too low relative to the stock price. An arbitrageur can

1. Short one share spot, investing the proceeds of the short sale at 5% per annum for three months.
2. Take a long position in a three-month forward contract on one share.

The proceeds of the short sale (i.e., \$40) grow to $40e^{0.05\times 3/12} = \$40.50$. At the end of the three months, the arbitrageur pays \$39 and takes delivery of the share under the terms of the forward contract. The share is used to close out the short position. The arbitrageur therefore makes a net profit at the end of the three-month period of

$$\$40.50 - \$39.00 = \$1.50$$

take a long position in a three-month forward contract. The proceeds of the short sale grow to

$$40e^{0.05\times 3/12}$$

or \$40.50 in three months. At the end of the three months, the arbitrageur pays \$39, takes delivery of the share under the terms of the forward contract, and uses it to close out the short position. A net gain of

$$\$40.50 - \$39.00 = \$1.50$$

is therefore made at the end of the three months. This trading strategy is summarized in Table 3.4.

Under what circumstances do arbitrage opportunities such as those in Tables 3.3 and 3.4 not exist? The arbitrage in Table 3.3 works when the forward price is greater than \$40.50. The arbitrage in Table 3.4 works when the forward price is less than \$40.50. We deduce that for there to be no arbitrage the forward price must be exactly \$40.50.

A Generalization

To generalize this example, we consider a forward contract on an investment asset with price S_0 that provides no income. Using our notation, T is the time to maturity, r is the risk-free rate, and F_0 is the forward price. The relationship between F_0 and S_0 is

$$F_0 = S_0 e^{rT} \tag{3.5}$$

If $F_0 > S_0 e^{rT}$, arbitrageurs can buy the asset and short forward contracts on the asset. If $F_0 < S_0 e^{rT}$, they can short the asset and buy forward contracts on it.[2] In our example,

[2] For another way of seeing that equation (3.5) is correct, consider the following strategy: buy one unit of the asset and enter into a short forward contract to sell it for F_0 at time T. This costs S_0 and is certain to lead to a cash inflow of F_0 at time T. S_0 must therefore equal the present value of F_0; that is, $S_0 = F_0 e^{-rT}$, or equivalently $F_0 = S_0 e^{rT}$.

$S_0 = 40$, $r = 0.05$, and $T = 0.25$ so that equation (3.5) gives

$$F_0 = 40e^{0.05\times0.25} = \$40.50$$

which is in agreement with our earlier calculations.

Example

Consider a four-month forward contract to buy a zero-coupon bond that will mature one year from today. The current price of the bond is \$930. (Because the bond will have eight months to go when the forward contract matures, we can regard the contract as on an eight-month zero-coupon bond.) We assume that the four-month risk-free rate of interest (continuously compounded) is 6% per annum. Because zero-coupon bonds provide no income, we can use equation (3.5) with $T = 4/12$, $r = 0.06$, and $S_0 = 930$. The forward price, F_0, is given by

$$F_0 = 930e^{0.06\times4/12} = \$948.79$$

This would be the delivery price in a contract negotiated today.

What If Short Sales Are Not Possible?

Short sales are not possible for all investment assets. As it happens, this does not matter. To derive equation (3.5) we do not need to be able to short the asset. All that we require is that there be a significant number of people who hold the asset purely for investment (and by definition this is always true of an investment asset). If the forward price is too low they will find it attractive to sell the asset and take a long position in a forward contract.

Suppose the underlying asset is gold and assume no storage costs. If $F_0 > S_0e^{rT}$ an investor can adopt the following strategy:

1. Borrow S_0 dollars at an interest rate r for T years.
2. Buy one ounce of gold.
3. Short a forward contract on one ounce of gold.

At time T one ounce of gold is sold for F_0. An amount S_0e^{rT} is required to repay the loan at this time and the investor makes a profit of $F_0 - S_0e^{rT}$.

Suppose next that $F_0 < S_0e^{rT}$. In this case an investor who owns one ounce of gold can

1. Sell the gold for S_0.
2. Invest the proceeds at interest rate r for time T.
3. Take a long position in a forward contract on one ounce of gold.

At time T the cash invested has grown to S_0e^{rT}. The gold is repurchased for F_0 and the investor makes a profit of $S_0e^{rT} - F_0$ relative to the position the investor would have been in if the gold had been kept.

As in the non-dividend-paying stock example considered earlier, we can expect the forward price to adjust so that neither of the two arbitrage opportunities we have considered exists. This means that the relationship in equation (3.5) must hold.

Table 3.5 Arbitrage opportunity when forward price of a coupon-bearing bond is too high

From the Trader's Desk

The forward price of a bond for a contract with a delivery date in one year is \$930. The current spot price is \$900. Coupon payments of \$40 are expected in six months and one year. The six-month and one-year risk-free interest rates are 9% per annum and 10% per annum, respectively.

Opportunity

The forward price is too high. An arbitrageur can

1. Borrow \$900 to buy one bond spot.
2. Short a forward contract on one bond.

The \$900 loan is made up of \$38.24 borrowed at 9% per annum for six months and \$861.76 borrowed at 10% per annum for one year. The first coupon payment of \$40 is exactly sufficient to repay interest and principal on the \$38.24. At the end of one year, the second coupon of \$40 is received, \$930 is received for the bond under the terms of the forward contract, and \$952.39 is required to pay principal and interest on the \$861.76. The net profit is, therefore,

$$\$40.00 + \$930.00 - \$952.39 = \$17.61$$

3.6 KNOWN INCOME

In this section we consider a forward contract on an investment asset that will provide a perfectly predictable cash income to the holder. Examples are stocks paying known dividends and coupon-bearing bonds. We adopt the same approach as in the previous section. We first look at a numerical example and then review the formal arguments.

Illustration

Consider a long forward contract to purchase a coupon-bearing bond whose current price is \$900. We will suppose that the forward contract matures in one year and the bond matures in five years, so that the forward contract is a contract to purchase a four-year bond in one year. We will also suppose that coupon payments of \$40 are expected after 6 months and 12 months, with the second coupon payment being immediately prior to the delivery date in the forward contract. We assume the six-month and one-year risk-free interest rates (continuously compounded) are 9% per annum and 10% per annum, respectively.

Suppose first that the forward price is relatively high at \$930. An arbitrageur can borrow \$900 to buy the bond and short a forward contract. The first coupon payment has a present value of $40e^{-0.09\times0.5} = \$38.24$. Of the \$900, \$38.24 is therefore borrowed at 9% per annum for six months so that it can be repaid with the first coupon payment. The remaining \$861.76 is borrowed at 10% per annum for one year. The amount owing at the end of the year is $861.76e^{0.1\times1} = \$952.39$. The second coupon provides \$40 toward this amount, and \$930 is received for the bond under the terms of the forward contract. The arbitrageur therefore makes a net profit of

$$\$40 + \$930 - \$952.39 = \$17.61$$

This strategy is summarized in Table 3.5.

Table 3.6 Opportunity when the forward price of a coupon-bearing bond is too low

From the Trader's Desk

The forward price of a bond for a contract with delivery date in one year is \$905. The current spot price is \$900. Coupon payments of \$40 are expected in six months and one year. The six-month and one-year risk-free interest rates are 9% per annum and 10% per annum, respectively.

Opportunity

The futures price is too low. An investor who holds the bond can

1. Sell one bond.
2. Enter into a long forward contract to repurchase the bond in one year.

Of the \$900 realized from selling the bond, \$38.24 is invested for six months at 9% per annum and \$861.76 is invested for one year at 10% per annum. This strategy produces a cash flow of \$40 at the six-month point and a cash flow of \$952.39 at the one-year point. The \$40 replaces the coupon that would have been received on the bond at the six-month point. Of the \$952.39, \$40 replaces the coupon that would have been received on the bond at the one-year point. Under the terms of the forward contract, the bond is repurchased for \$905. The strategy of selling the bond spot and buying it back forward is, therefore,

$$\$952.39 - \$40.00 - \$905.00 = \$7.39$$

more profitable than simply holding the bond for the year.

Suppose next that the forward price is relatively low at \$905. An investor who holds the bond can sell it and enter into a long forward contract. Of the \$900 realized from selling the bond, \$38.24 is invested for 6 months at 9% per annum so that it grows into an amount sufficient to equal the coupon that would have been paid on the bond. The remaining \$861.76 is invested for 12 months at 10% per annum and grows to \$952.39. Of this sum, \$40 is used to replace the coupon that would have been received on the bond, and \$905 is paid under the terms of the forward contract to replace the bond in the investor's portfolio. The investor therefore gains

$$\$952.39 - \$40.00 - \$905.00 = \$7.39$$

relative to the situation the investor would have been in by keeping the bond. This strategy is summarized in Table 3.6.

The strategy in Table 3.5 produces a profit when the forward price is greater than \$912.39, whereas the strategy in Table 3.6 produces a profit when the forward price is less than \$912.39. It follows that if there are no arbitrage opportunities, the forward price must be \$912.39.

A Generalization

We can generalize from this example to argue that when an investment asset will provide income with a present value of I during the life of a forward contract

$$F_0 = (S_0 - I)e^{rT} \tag{3.6}$$

In our example, $S_0 = 900.00$, $I = 40e^{-0.09\times0.5} + 40e^{-0.10\times1} = 74.433$, $r = 0.1$, and $T = 1$

so that

$$F_0 = (900.00 - 74.433)e^{0.1\times 1} = \$912.39$$

This is in agreement with our earlier calculation. Equation (3.6) applies to any asset that provides a known cash income.

If $F_0 > (S_0 - I)e^{rT}$, an arbitrageur can lock in a profit by buying the asset and shorting a forward contract on the asset. If $F_0 < (S_0 - I)e^{rT}$ an arbitrageur can lock in a profit by shorting the asset and taking a long position in a forward contract. If short sales are not possible, investors who own the asset will find it profitable to sell the asset and enter into long forward contracts.[3]

Example

Consider a 10-month forward contract on a stock with a price of \$50. We assume that the risk-free rate of interest (continuously compounded) is 8% per annum for all maturities. We also assume that dividends of \$0.75 per share are expected after three months, six months, and nine months. The present value of the dividends, I, is given by

$$I = 0.75e^{-0.08\times 3/12} + 0.75e^{-0.08\times 6/12} + 0.75e^{-0.08\times 9/12} = 2.162$$

The variable T is 10 months so that the forward price, F_0, from equation (3.6), is given by

$$F_0 = (50 - 2.162)e^{0.08\times 10/12} = \$51.14$$

If the forward price were less than this, an arbitrageur would short the stock spot and buy forward contracts. If the forward price were greater than this, an arbitrageur would short forward contracts and buy the stock spot.

3.7 KNOWN YIELD

We now consider the situation where the asset underlying a forward contract provides a known yield rather than a known cash income. This means that the income is known when expressed as a percent of the asset's price at the time the income is paid. Suppose that an asset is expected to provide a yield of 5% per annum. This could mean that income equal to 5% of the asset price is paid once a year. (The yield would then be 5% with annual compounding.) It could mean that income equal to 2.5% of the asset price is paid twice a year. (The yield would then be 5% per annum with semiannual compounding.) In Section 3.3 we explained that we will normally measure interest rates with continuous compounding. Similarly we will normally measure yields with continuous compounding. Formulas for translating a yield measured with one compounding frequency to a yield measured with another compounding frequency are the same as those given for interest rates in Section 3.3.

Define q as the average yield per annum on an asset during the life of a forward

[3] For another way of seeing that equation (3.6) is correct, consider the following strategy: buy one unit of the asset and enter into a short forward contract to sell it for F_0 at time T. This costs S_0 and is certain to lead to a cash inflow of F_0 at time T and income with a present value of I. The initial outflow is S_0. The present value of the inflows is $I + F_0e^{-rT}$. Hence $S_0 = I + F_0e^{-rT}$, or equivalently $F_0 = (S_0 - I)e^{rT}$.

contract. It can be shown (see Problem 3.22) that

$$F_0 = S_0 e^{(r-q)T} \tag{3.7}$$

Example

Consider a six-month forward contract on an asset that is expected to provide income equal to 2% of the asset once during a six-month period. The risk-free rate of interest (with continuous compounding) is 10% per annum. The asset price is \$25. In this case $S_0 = 25$, $r = 0.10$, and $T = 0.5$. The yield is 4% per annum with semiannual compounding. From equation (3.3) this is 3.96% per annum with continuous compounding. It follows that $q = 0.0396$ so that from equation (3.7) the forward price F_0 is given by

$$F_0 = 25e^{(0.10-0.0396)\times 0.5} = \$25.77$$

3.8 VALUING FORWARD CONTRACTS

The value of a forward contract at the time it is first entered into is zero. At a later stage it may prove to have a positive or negative value. Using the notation introduced earlier, we suppose F_0 is the current forward price for contract that was negotiated some time ago, the delivery date is in T years, and r is the T-year risk-free interest rate. We also define

K: Delivery price in the contract

f: Value of the forward contract today

A general result, applicable to all forward contracts (both those on investment assets and those on consumption assets), is

$$f = (F_0 - K)e^{-rT} \tag{3.8}$$

When the forward contract is first negotiated K is set equal to F_0 and $f = 0$. As time passes, both the forward price, F_0, and the value of the forward contract, f, change.

To see why equation (3.8) is correct, we compare a long forward contract that has a delivery price of F_0 with an otherwise identical long forward contract that has a delivery price of K. The difference between the two is only in the amount that will be paid for the underlying asset at time T. Under the first contract this amount is F_0; under the second contract it is K. A cash outflow difference of $F_0 - K$ at time T translates to a difference of $(F_0 - K)e^{-rT}$ today. The contract with a delivery price F_0 is therefore less valuable than the contract with delivery price K by an amount $(F_0 - K)e^{-rT}$. The value of the contract that has a delivery price of F_0 is by definition zero. It follows that the value of the contract with a delivery price of K is $(F_0 - K)e^{-rT}$. This proves equation (3.8). Similarly, the value of a short forward contract with delivery price K is

$$(K - F_0)e^{-rT}$$

Example

A long forward contract on a non-dividend-paying stock was entered into some time ago. It currently has six months to maturity. The risk-free rate of interest (with continuous compounding) is 10% per annum, the stock price is $25, and the delivery price is $24. In this case $S_0 = 25$, $r = 0.10$, $T = 0.5$, and $K = 24$. From equation (3.5) the six-month forward price, F_0, is given by

$$F_0 = 25e^{0.1 \times 0.5} = \$26.28$$

From equation (3.8), the value of the forward contract is

$$f = (26.28 - 24)e^{-0.1 \times 0.5} = \$2.17$$

Equation (3.8) shows that we can value a long forward contract on an asset by making the assumption that the price of the asset at the maturity of the forward contract equals the forward price F_0. To see this, note that when we make the assumption, a long forward contract provides a payoff at time T of $F_0 - K$. This has a present value of $(F_0 - K)e^{-rT}$, which is the value of f in equation (3.8). Similarly, we can value a short forward contract on the asset by assuming that the current forward price of the asset is realized.

Using equation (3.8) in conjunction with (3.5) gives the following expression for the value of a forward contract on an investment asset that provides no income:

$$f = S_0 - Ke^{-rT} \tag{3.9}$$

Similarly, using equation (3.8) in conjunction with (3.6) gives the following expression for the value of a long forward contract on an investment asset that provides a known income with present value I:

$$f = S_0 - I - Ke^{-rT} \tag{3.10}$$

Finally, using equation (3.8) in conjunction with (3.7) gives the following expression for the value of a long forward contract on an investment asset that provides a known yield at rate q:

$$f = S_0e^{-qT} - Ke^{-rT} \tag{3.11}$$

3.9 ARE FORWARD PRICES AND FUTURES PRICES EQUAL?

The Appendix at the end of this chapter provides an arbitrage argument to show that when the risk-free interest rate is constant and the same for all maturities, the forward price for a contract with a certain delivery date is the same as the futures price for a contract with that delivery date. The argument in the Appendix can be extended to cover situations where the interest rate is a known function of time.

When interest rates vary unpredictably (as they do in the real world), forward and futures prices are in theory no longer the same. The proof of the relationship between the two is beyond the scope of this book. However, we can get a sense of the nature of the relationship by considering the situation where the price of the underlying asset, S, is strongly positively correlated with interest rates. When S increases, an investor who holds a long futures position makes an immediate gain because of the daily settlement procedure. The positive correlation indicates that it is likely that interest rates have also increased. The gain will therefore tend to be invested at a higher than average rate of interest. Similarly, when S decreases, the investor will incur an immediate loss. This loss will tend to be financed at a lower than average rate of interest. An investor holding a forward contract rather than a futures contract is not affected in this way by interest-rate movements. It follows that a long futures contract will be more attractive than a similar long forward contract. Hence, when S is strongly positively correlated with interest rates, futures prices will tend to be higher than forward prices. When S is strongly negatively correlated with interest rates, a similar argument shows that forward prices will tend to be higher than futures prices.

The theoretical differences between forward and futures prices for contracts that last only a few months are in most circumstances sufficiently small to be ignored. In practice, there are a number of factors not reflected in theoretical models that may cause forward and futures prices to be different. These include taxes, transactions costs, and the treatment of margins. The risk that the counterparty will default is generally less in the case of a futures contract because of the role of the exchange clearinghouse. Also, in some instances, futures contracts are more liquid and easier to trade than forward contracts. Despite all these points, for most purposes it is reasonable to assume that forward and futures prices are the same. This is the assumption we will usually make in this book. We will use the symbol F_0 to represent both the futures price and the forward price of an asset.

As the life of a futures contract increases, the differences between forward and futures contracts are liable to become significant. It is then dangerous to assume that forward and futures prices are perfect substitutes for each other. This point is particularly relevant to Eurodollar futures contracts that have maturities as long as 10 years. These contracts are covered in Chapter 5.

Empirical Research

Some empirical research that has been carried out comparing forward and futures contracts is listed at the end of this chapter. Cornell and Reinganum studied forward and futures prices on the British pound, Canadian dollar, German mark, Japanese yen, and Swiss franc between 1974 and 1979. They found very few statistically significant differences between the two sets of prices. Their results were confirmed by Park and Chen, who as part of their study looked at the British pound, German mark, Japanese yen, and Swiss franc between 1977 and 1981.

French studied copper and silver during the period from 1968 to 1980. The results for silver show that the futures price and the forward price are significantly different (at the 5% confidence level), with the futures price generally above the forward price. The results for copper are less clear-cut. Park and Chen looked at gold, silver, silver coin, platinum, copper, and plywood between 1977 and 1981. Their results are similar to those of French for silver. The forward and futures prices are significantly different, with the futures price above the forward price. Rendleman and Carabini studied the Treasury bill market between 1976 and 1978. They also found statistically significant differences between futures and forward prices. In all these studies, it seems likely that the differences observed are due to the factors mentioned in the previous section (taxes, transactions costs, and so on).

3.10 STOCK INDEX FUTURES

A *stock index* tracks changes in the value of a hypothetical portfolio of stocks. The weight of a stock in the portfolio equals the proportion of the portfolio invested in the stock. The percentage increase in the stock index over a small interval of time is set equal to the percentage increase in the value of the hypothetical portfolio. Dividends are usually not included in the calculation so that the index tracks the capital gain/loss from investing in the portfolio.[4]

If the hypothetical portfolio of stocks remains fixed, the weights assigned to individual stocks in the portfolio do not remain fixed. When the price of one particular stock in the portfolio rises more sharply than others, more weight is automatically given to that stock. Some indices are constructed from a hypothetical portfolio consisting of one of each of a number of stocks. The weights assigned to the stocks are then proportional to their market prices, with adjustments being made when there are stock splits. Other indices are constructed so that weights are proportional to market capitalization (stock price × number of shares outstanding). The underlying portfolio is then automatically adjusted to reflect stock splits, stock dividends, and new equity issues.

Stock Indices

Table 3.7 shows futures prices for contracts on a number of different stock indices as they were reported in the *Wall Street Journal* of March 16, 2001. The prices refer to the close of trading on March 15, 2001.

The *Dow Jones Industrial Average* is based on a portfolio consisting of 30 blue-chip stocks in the United States. The weights given to the stocks are proportional to their prices. One futures contract, traded on the Chicago Board of Trade, is on $10 times the index.

The *Standard & Poor's 500 (S&P 500) Index* is based on a portfolio of 500 different stocks: 400 industrials, 40 utilities, 20 transportation companies, and 40 financial institutions. The weights of the stocks in the portfolio at any given time are proportional to their market capitalizations. This index accounts for 80% of the market capitalization of all the stocks listed on the New York Stock Exchange. The Chicago Mercantile Exchange (CME) trades two contracts on the S&P 500. One is on $250 times the index; the other (the Mini S&P 500 contract) is on $50 times the index. The *Standard & Poor's MidCap 400 Index* is similar to the S&P 500, but based on a portfolio of 400 stocks that have somewhat lower market capitalizations.

The *Nikkei 225 Stock Average* is based on a portfolio of 225 of the largest stocks trading on the Tokyo Stock Exchange. Stocks are weighted according to their prices. One futures contract (traded on the CME) is on $5 times the index.

The *Nasdaq 100* is based on 100 stocks using the National Association of Securities Dealers Automatic Quotations Service. The CME trades two contracts. One is on $100 times the index; the other (the Mini Nasdaq 100 contract) is on $20 times the index.

In the GSCI index futures contract shown in Table 3.7, the underlying asset is the *Goldman Sachs Commodity Index*. This is not a stock index. It is a broadly based index of commodity prices. All the major commodity groups, such as energy, livestock, grains and oilseeds, food and fiber, and metals, are represented in the GSCI. Studies by Goldman Sachs have shown that the GSCI is negatively related to the S&P 500 index, with the correlation being in the range −0.30 to −0.40.

The *Russell 2000* index is an index of small stocks in the United States. The *U.S. dollar index* is a trade-weighted index of the values of six foreign currencies (the euro, yen, pound, Canadian dollar, Swedish krona, and Swiss franc). The *Share Price Index* is

[4] An exception to this is a *total return index*. This is calculated by assuming that dividends on the hypothetical portfolio are reinvested in the portfolio.

Table 3.7 Stock index futures quotes from the *Wall Street Journal* on March 16, 2001

INDEX

DJ Industrial Average (CBOT)-$10 times average									
Mar	10095	10115	9980	10020	+	12	11640	9880	8,023
June	10170	10200	10060	10105	+	10	11795	9980	24,367
Sept	10260	10260	10160	10200	+	7	11330	10095	294
Est vol 19,000; vol Wed 45,038; open int 32,716, +1,471.									
Idx prl: Hi 10097.73; Lo 9980.85; Close 10031.28, +57.82.									
S&P 500 Index (CME)-$250 times index									
Mar	118000	118200	116950	117330	+	410	164260	115500	112,346
June	118080	119450	117450	118470	+	390	166660	116550	455,531
Sept	119500	120680	119300	119640	+	360	169060	117820	2,635
Dec	120700	121780	120500	120740	+	360	171460	118920	969
Mr02	....	....	....	121790	+	330	173860	120070	409
June	123500	124200	122500	123040	+	380	170550	121320	459
Est vol 113,148; vol Wed 209,907; open int 572,426, +8,748.									
Idx prl: Hi 1182.04; Lo 1166.71; Close 1173.56, +6.85.									
Mini S&P 500 (CME)-$50 times index									
Mar	116950	118200	116400	117325	+	400	150000	115400	58,162
Vol Wed 183,181; open int 99,765, −2,471.									
S&P Midcap 400 (CME)-$500 times index									
Mar	475.00	477.00	471.00	471.55	−	1.90	564.00	450.50	4,347
June	480.00	484.00	475.50	476.15	−	2.10	571.00	475.00	15,905
Est vol 3,128; vol Wed 4,403; open int 20,262, −48.									
Idx prl: Hi 479.23; Lo 471.25; Close 471.25, −2.34.									
Nikkei 225 Stock Average (CME)-$5 times index									
June	12080.	12175.	12030.	12130.	+	735	17730.	11255.	16,825
Est vol 1,233; vol Wed 2,341; open int 16,846, +81.									
Idx prl: Hi 12152.83; Lo 11433.88; Close 12152.83, +309.24.									
Nasdaq 100 (CME)-$100 times index									
Mar	181400	181800	168000	169250	−	6000	424150	168000	21,284
June	178000	184100	170800	171500	−	6100	396100	169800	47,728
Est vol 25,437; vol Wed 41,468; open int 69,062, −3,889.									
Idx prl: Hi 1813.68; Lo 1697.61; Close 1697.92, −47.16.									
Mini Nasdaq 100 (CME)-$20 times index									
Mar	1761.5	1813.0	1683.0	1692.5	−	60.0	3850.0	1676.0	65,740
Vol Wed 146,423; open int 104,082, −771.									
GSCI (CME)-$250 times nearby index									
Mar	217.00	217.00	214.40	na		na	250.00	214.40	2,503
Apr	216.90	217.00	214.40	215.50	−	.50	237.50	214.40	14,551
Est vol 3,241; vol Wed 3,807; open int 17,055, +157.									
Idx prl: Hi 217.26; Lo 214.34; Close 215.58, −.52.									
Russell 2000 (CME)-$500 times index									
Mar	458.00	458.00	450.00	452.00	−	.10	603.10	445.50	4,407
June	461.00	463.00	454.50	456.00	−	.50	574.65	454.50	16,083
Est vol 4,190; vol Wed 4,632; open int 20,490, −40.									
Idx prl: Hi 457.96; Lo 451.71; Close 452.16, −1.53.									
U.S. Dollar Index (NYBOT)-$1,000 times USDX									
Mar	114.10	115.10	114.10	114.72	+	.96	118.72	108.04	828
June	113.73	115.38	113.55	114.87	+	1.00	118.13	108.18	5,297
Est vol 3,100; vol Wed 5,031; open int 8,133, +485.									
Idx prl: Hi 115.19; Lo 113.43; Close 114.71, +1.01.									
Share Price Index (SFE)									
A $25 times index									
Mar	3261.0	3261.0	3197.0	3247.0	−	14.0	3395.0	3045.0	163,238
June	3284.0	3288.0	3228.0	3273.0	−	14.0	3410.0	3080.0	16,992
Sept	....	....	....	3289.0	−	16.0	3450.0	3303.0	1,442
Dec	....	....	....	3307.0	−	14.0	3435.0	3318.0	528
Est vol 28,053; vol Wed 15,314; open int 182,234, +13,838.									
Index Hi 3263.9; Lo 3204.1; Close 3242.9, −21.0.									
CAC-40 Stock Index (MATIF)-Euro 10.00 x index									
Mar	5140.0	5207.5	5069.0	5176.0	+	50.0	7102.0	4489.0	370,341
Apr	5175.0	5196.5	5100.0	5195.0	+	49.0	6022.5	5032.0	12,519
May	5145.0	5180.5	5070.5	5167.0	+	50.0	5508.5	4976.5	75
June	5119.0	5154.0	5095.0	5156.0	+	51.0	7034.0	4973.0	14,742
Sept	....	....	....	5207.0	+	51.0	6013.5	4804.0	5,930
Dec	....	....	....	5261.0	+	52.0	6162.5	5892.5	913
Mr02	....	....	....	5314.0	+	51.0	....	....	2,600
Sept	....	....	....	5346.0	+	52.0	....	....	600
Est vol 117,659; vol Wed 94,843; open int 407,720, +21,514.									
DAX-30 German Stock Index (EUREX)									
Euro 25 per DAX index pt.									
Mar	5850.0	5887.5	5768.5	5856.0	+	56.0	7699.0	5665.0	190,163
June	5876.5	5912.0	5793.0	5882.5	+	61.5	7364.0	5690.0	293,541
Sept	5888.0	5959.5	5888.0	5945.5	+	64.0	6952.5	5786.0	2,844
Vol Thu 125,542; open int 486,548, +49,245.									
Index Hi 5889.95 Lo 5767.06 Close 5889.95, +95.83.									
FT-SE 100 Index (LIFFE)-£10 per index point									
Mar	5667.5	5718.0	5596.5	5669.0	+	45.0	6620.0	5463.0	71,880
June	5710.0	5751.5	5640.0	5709.0	+	41.5	6398.0	618.0	260,404
Sept	....	....	....	5749.0	+	39.0	6436.0	5966.5	6,750
Est vol 113,700; vol Wed 151,724; open int 339,034, +10,145.									
DJ Euro Stoxx 50 Index (EUREX)-Euro 10.00 x index									
Mar	4140.0	4189.0	4086.0	4180.0	+	63.0	5536.0	3998.0	340,951
June	4135.0	4174.0	4070.0	4166.0	+	62.0	5232.0	3984.0	434,041
Sept	4195.0	4195.0	4191.0	4202.0	+	66.0	4913.0	4058.0	24,311
Vol Thu 361,073; open int 799,303, +44,714.									
Index Hi 4200.08; Lo 4088.84; Close 4200.08, +76.11.									
DJ Stoxx 50 Index (EUREX)-Euro 10.00 x index									
Mar	3945.0	4000.0	3903.0	3991.0	+	51.0	5159.0	3803.0	13,901
June	3940.0	4008.0	3905.0	3999.0	+	52.0	5050.0	3811.0	14,427
Vol Thu 6,794; open int 28,334, +894.									
Index Hi 4032.16; Lo 3908.72; Close 4032.98, +91.22.									

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the All Ordinaries Share Price Index, a broadly based index of Australian stocks. The *CAC-40 Index* is based on 40 large stocks trading in France. The *DAX-30 Index* is based on 30 stocks trading in Germany. The *FT-SE 100 Index* is based on a portfolio of 100 major U.K. stocks listed on the London Stock Exchange. The *DJ Euro Stoxx 50 Index* and the *DJ Stoxx 50 Index* are two different indices of blue chip European stocks compiled by Dow Jones and its European partners. The futures contracts on these indices trade on Eurex and are on 10 times the values of the indices measured in euros.

As we mentioned in Chapter 2, futures contracts on stock indices are settled in cash, not by delivery of the underlying asset. All contracts are marked to market at either the opening price or the closing price of the index on the last trading day, and the positions are then deemed to be closed. For example, contracts on the S&P 500 are closed out at the opening S&P 500 index on the third Friday of the delivery month. Trading in the contracts continues until 8:30 a.m. on that Friday.

Futures Prices of Stock Indices

A stock index can be regarded as the price of an investment asset that pays dividends. The investment asset is the portfolio of stocks underlying the index, and the dividends paid by the investment asset are the dividends that would be received by the holder of this portfolio. It is usually assumed that the dividends provide a known yield rather than a known cash income. If q is the dividend yield rate, equation (3.7) gives the futures price, F_0, as

$$F_0 = S_0 e^{(r-q)T} \tag{3.12}$$

Example

Consider a three-month futures contract on the S&P 500. Suppose that the stocks underlying the index provide a dividend yield of 1% per annum, that the current value of the index is 400, and that the continuously compounded risk-free interest rate is 6% per annum. In this case, $r = 0.06$, $S_0 = 400$, $T = 0.25$, and $q = 0.01$. Hence, the futures price, F_0, is given by

$$F_0 = 400e^{(0.06-0.01)\times 0.25} = \$405.03$$

In practice, the dividend yield on the portfolio underlying an index varies week by week throughout the year. For example, a large proportion of the dividends on the NYSE stocks are paid in the first week of February, May, August, and November each year. The chosen value of q should represent the average annualized dividend yield during the life of the contract. The dividends used for estimating q should be those for which the ex-dividend date is during the life of the futures contract. Looking at Table 3.7, we see that the futures prices for the S&P 500 Index appear to be increasing with the maturity of the futures contract at about 3.8% per annum. This corresponds to the situation where the risk-free interest rate exceeds the dividend yield by about 3.8% per annum.

Index Arbitrage

If $F_0 > S_0 e^{(r-q)T}$, profits can be made by buying spot (i.e., for immediate delivery) the stocks underlying the index and shorting futures contracts. If $F_0 < S_0 e^{(r-q)T}$, profits can be made by doing the reverse—that is, shorting or selling the stocks underlying the index and taking a long position in futures contracts. These strategies are known as *index arbitrage*. When $F_0 < S_0 e^{(r-q)T}$, index arbitrage is often done by a pension fund that owns an indexed portfolio of stocks. When $F_0 > S_0 e^{(r-q)T}$, it is often done by a corporation holding short-term money market investments. For indices involving many stocks, index arbitrage is sometimes accomplished by trading a relatively small representative sample of stocks whose movements closely mirror those of the index. Often index arbitrage is implemented through *program trading*, with a computer system being used to generate the trades.

October 1987

To do index arbitrage a trader must be able to trade both the index futures contract and the portfolio of stocks underlying the index very quickly at the prices quoted in the market. In normal market conditions this is possible using program trading, and F_0 is very close to $S_0 e^{(r-q)T}$. Examples of days when the market was anything but normal are October 19 and 20 of 1987. On what is termed "Black Monday," October 19, 1987, the market fell by more than 20%, and the 604 million shares traded on the New York Stock Exchange easily exceeded all previous records. The exchange's systems were overloaded, and if you placed an order to buy or sell a share on that day there could be a delay of up to two hours before your order was executed. For most of the day, futures prices were at a significant discount to the underlying index. For example, at the close of trading the S&P 500 Index was at 225.06 (down 57.88 on the day), whereas the futures price for December delivery on the S&P 500 was 201.50 (down 80.75 on the day). This was largely because the delays in processing orders made index arbitrage impossible. On the next day, Tuesday, October 20, 1987, the New York Stock Exchange placed temporary restrictions on the way in which program trading could be done. This also made index arbitrage very difficult, and the breakdown of the traditional linkage between stock indices and stock index futures continued. At one point the futures price for the December contract was 18% less than the S&P 500 Index. However, after a few days the market returned to normal, and the activities of arbitrageurs ensured that equation (3.12) governed the relationship between futures and spot prices of indices.

The Nikkei Futures Contract

Equation (3.12) does not apply to the futures contract on the Nikkei 225. The reason is quite subtle. When S is the value of the Nikkei 225 Index, it is the value of a portfolio measured in yen. The variable underlying the CME futures contract on the Nikkei 225 has a *dollar value* of $5S$. In other words, the futures contract takes a variable that is measured in yen and treats it as though it is dollars. We cannot invest in a portfolio whose value will always be $5S$ dollars. The best we can do is to invest in one that is always worth $5S$ yen or in one that is always worth $5QS$ dollars, where Q is the dollar value of one yen. The arbitrage arguments that have been used in this chapter require the spot price underlying the futures price to be the price of an asset that can be traded by investors. The arguments are therefore not exactly correct for the Nikkei 225 contract.

3.11 FORWARD AND FUTURES CONTRACTS ON CURRENCIES

We now move on to consider forward and futures foreign currency contracts. The underlying asset in such contracts is a certain number of units of the foreign currency. We will, therefore, define the variable S_0 as the current spot price in dollars of one unit of the foreign currency and F_0 as the forward or futures price in dollars of one unit of the foreign currency. This is consistent with the way we have defined S_0 and F_0 for other assets underlying forward and futures contracts. However, as mentioned in Chapter 2, it does not necessarily correspond to the way spot and forward exchange rates are quoted. For major exchange rates other than the British pound, euro, Australian dollar, and New Zealand dollar, a spot or forward exchange rate is normally quoted as the number of units of the currency that are equivalent to one dollar.

A foreign currency has the property that the holder of the currency can earn interest at the risk-free interest rate prevailing in the foreign country. For example, the holder can invest the currency in a foreign-denominated bond. We define r_f as the value of the foreign risk-free interest rate when money is invested for time T. As before, r is the domestic risk-free rate when money is invested for this period of time.

The relationship between F_0 and S_0 is

$$F_0 = S_0 e^{(r-r_f)T} \tag{3.13}$$

This is the well-known interest rate parity relationship from international finance. To see that it must be true, we suppose that the two-year interest rates in Australia and the United States are 5% and 7% respectively, and the spot exchange rate between the

Table 3.8 Foreign exchange futures quotes from the *Wall Street Journal* on March 16, 2001

CURRENCY

Japan Yen (CME)-12.5 million yen; $ per yen (.00)								
Mar	.8270	.8297	.8160	.8174	− .0094	1.0300	.8160	41,711
June	.8358	.8398	.8256	.8270	− .0095	1.0219	.8256	87,632
Sept	.8475	.8475	.8345	.8363	− .0097	1.0050	.8345	579
Dec	.8450	.8450	.8450	.8455	− .0099	.9880	.8450	431
Est vol 23,771; vol Wed 55,559; open int 130,445, +6,474.								
Deutschemark (CME)-125,000 marks; $ per mark								
Mar	.4652	.4652	.4581	.4606	− .0061	.4925	.4225	236
June	.4660	.4661	.4596	.4607	− .0058	.4900	.4596	229
Est vol 217; vol Wed 161; open int 467, −172.								
Canadian Dollar (CME)-100,000 dlrs.; $ per Can $								
Mar	.6425	.6431	.6401	.6405	− .0018	.7040	.6401	20,442
June	.6427	.6434	.6404	.6408	− .0018	.6990	.6404	56,601
Sept	.6425	.6436	.6405	.6412	− .0018	.6906	.6405	2,630
Dec	.6440	.6440	.6417	.6416	− .0018	.6825	.6417	1,286
Est vol 8,574; vol Wed 27,233; open int 81,108, +998.								
British Pound (CME)-62,500 pds.; $ per pound								
Mar	1.4448	1.4488	1.4340	1.4392	− .0064	1.6050	1.4010	14,833
June	1.4444	1.4478	1.4330	1.4374	− .0064	1.5304	1.4060	23,641
Est vol 7,361; vol Wed 15,385; open int 38,510, −2,720.								
Swiss Franc (CME)-125,000 francs; $ per franc								
Mar	.5910	.5910	.5828	.5858	− .0051	.6326	.5541	20,680
June	.5885	.5951	.5842	.5879	− .0052	.6358	.5585	32,622
Est vol 14,447; vol Wed 34,342; open int 53,337, +4,514.								
Australian Dollar (CME)-100,000 dlrs.; $ per A.$								
Mar	.4960	.4960	.4908	.4925	− .0019	.6390	.4908	18,376
June	.4956	.4971	.4898	.4924	− .0020	.6083	.4898	23,621
Sept	.4942	.4942	.4917	.4923	− .0021	.5622	.4917	264
Est vol 3,212; vol Wed 8,578; open int 42,329, +313.								
Mexican Peso (CME)-500,000 new Mex. peso, $ per MP								
Mar	.10438	.10445	.10395	.10403	− 00017	.10453	.09120	13,248
Apr	....	....	....	.10298	− 00027	.10353	.09730	390
May	....	....	....	.10198	− 00027	.10180	.09900	848
June	.10155	.10170	.10095	.10108	− 00027	.10170	.09070	20,061
Aug	....	....	....	.09908	− 00027	.09800	.09800	100
Sept	....	....	....	.09815	− 00027	.09880	.09300	2,693
Euro FX (CME)-Euro 125,000; $ per Euro								
Mar	.9116	.9120	.8965	.9009	− .0089	.9999	.8333	38,657
June	.9121	.9130	.8960	.9010	− .0092	.9784	.8358	59,061
Sept	.9060	.9071	.8990	.9013	− .0093	.9634	.8379	1,178
Est vol 33,027; vol Wed 40,744; open int 99,063, −3,000.								

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Australian dollar (AUD) and the U.S. dollar (USD) is 0.6200 USD per AUD. From equation (3.13) the two-year forward exchange rate should be

$$0.62e^{(0.07-0.05)\times 2} = 0.6453$$

Suppose first the two-year forward exchange rate is less than this, say 0.6300. An arbitrageur can:

1. Borrow 1,000 AUD at 5% per annum for two years, convert to 620 USD and invest the USD at 7%. (Both rates are continuously compounded.)
2. Enter into a forward contract to buy 1,105.17 AUD for $1{,}105.17 \times 0.63 = 696.26$ USD.

The 620 USD that are invested at 7% grow to $620e^{0.07\times 2} = 713.17$ USD in two years. Of this, 696.26 USD are used to purchase 1,105.17 AUD under the terms of the forward contract. This is exactly enough to repay principal and interest on the 1,000 AUD that are borrowed ($1{,}000e^{0.05\times 2} = 1{,}105.17$). The strategy therefore gives rise to a riskless profit of $713.17 - 696.26 = 16.91$ USD. (If this does not sound very exciting, consider following a similar strategy where you borrow 100 million AUD!)

Suppose next that the two-year forward rate is 0.6600 (greater than the 0.6453 value given by equation (3.13)). An arbitrageur can:

1. Borrow 1,000 USD at 7% per annum for two years, convert to $1{,}000/0.6200 = 1{,}612.90$ AUD, and invest the AUD at 5%.
2. Enter into a forward contract to sell 1,782.53 AUD for $1{,}782.53 \times 0.66 = 1{,}176.47$ USD.

The 1,612.90 AUD that are invested at 5% grow to $1{,}612.90e^{0.05\times 2} = 1{,}782.53$ AUD in two years. The forward contract has the effect of converting this to 1,176.47 USD. The amount needed to payoff the USD borrowings is $1{,}000e^{0.07\times 2} = 1{,}150.27$ USD. The strategy therefore gives rise to a riskless profit of $1{,}176.47 - 1{,}150.27 = 26.20$ USD.

Table 3.8 shows futures prices on March 15, 2001, for a variety of different currency futures trading on the Chicago Mercantile Exchange. In the case of the Japanese yen, prices are expressed as the number of cents per unit of foreign currency. In the case of the other currencies, prices are expressed as the number of U.S. dollars per unit of foreign currency.

When the foreign interest rate is greater than the domestic interest rate ($r_f > r$), equation (3.13) shows that F_0 is always less than S_0 and that F_0 decreases as the time to maturity of the contract, T, increases. Similarly, when the domestic interest rate is greater than the foreign interest rate ($r > r_f$), equation (3.13) shows that F_0 is always greater than S_0 and that F_0 increases as T increases. On March 15, 2001, interest rates on the Japanese yen, Canadian dollar, and the euro were lower than the interest rate on the U.S. dollar. This corresponds to the $r > r_f$ situation and explains why futures prices for these currencies increase with maturity in Table 3.8. In Australia, Britain, and Mexico interest rates were higher than in the United States. This corresponds to the $r_f > r$ situation and explains why the futures price of the Mexican peso decreases with maturity.

Example

The futures price of the Japanese yen in Table 3.8 appears to be increasing at a rate of about 4.6% per annum with the maturity of the contract. The increase suggests that short-term interest rates were about 4.6% per annum higher in the United States than in Japan on March 15, 2001.

A Foreign Currency as an Asset Providing a Known Yield

Note that equation (3.13) is identical to equation (3.7) with q replaced by r_f. This is not a coincidence. A foreign currency can be regarded as an investment asset paying a known yield. The yield is the risk-free rate of interest in the foreign currency.

To understand this, suppose that the one-year interest rate on British pounds is 5% per annum. (For simplicity we assume that the interest rate is measured with annual compounding and interest is paid at the end of the year.) Consider a United States investor who buys 1 million pounds. The investor knows that £50,000 of interest will be earned at the end of one year. The value of this interest in dollars depends the exchange rate. If the exchange rate in one year is 1.5000, the interest is worth $75,000; if it is 1.4000, the interest is worth $70,000; and so on. The value in dollars of the interest earned is 5% of the value of the sterling investment. The 5% interest therefore represents a known yield to the United States investor on the sterling investment.

3.12 FUTURES ON COMMODITIES

We now move on to consider futures contracts on commodities. First we consider the impact of storage on the futures prices of commodities that are investment assets such as gold and silver.[5]

Storage Costs

Equation (3.5) shows that in the absence of storage costs the forward price of a commodity, such as gold or silver, that is an investment asset is given by

$$F_0 = S_0 e^{rT} \tag{3.14}$$

Storage costs can be regarded as negative income. If U is the present value of all the storage costs that will be incurred during the life of a forward contract, it follows from equation (3.6) that

$$F_0 = (S_0 + U)e^{rT} \tag{3.15}$$

Example

Consider a one-year futures contract on gold. Suppose that it costs \$2 per ounce per year to store gold, with the payment being made at the end of the year. Assume that the spot price is \$450 and the risk-free rate is 7% per annum for all maturities. This corresponds to $r = 0.07$, $S_0 = 450$, $T = 1$, and

$$U = 2e^{-0.07\times 1} = 1.865$$

From equation (3.15) the futures price, F_0, is given by

$$F_0 = (450 + 1.865)e^{0.07\times 1} = \$484.63$$

If $F_0 > 484.63$, an arbitrageur can buy gold and short one-year gold futures contracts to lock in a profit. If $F_0 < 484.63$, an investor who already owns gold can improve the return by selling the gold and buying gold futures contracts. Tables 3.9 and 3.10 illustrate these strategies for the situations where $F_0 = 500$ and $F_0 = 470$.

If the storage costs incurred at any time are proportional to the price of the commodity, they can be regarded as providing a negative yield. In this case, from equation (3.7),

$$F_0 = S_0 e^{(r+u)T} \tag{3.16}$$

where u is the storage costs per annum as a proportion of the spot price.

Consumption Commodities

For commodities that are consumption assets rather than investment assets, the arbitrage arguments used to determine futures prices need to be reviewed carefully.

[5] Recall that for an asset to be an investment asset, it need not be held solely for investment purposes. What is required is that some individuals hold it for investment purposes and that these individuals be prepared to sell their holdings and go long forward contracts, if the latter look more attractive. This explains why silver, although it has significant industrial uses, is an investment asset.

Table 3.9 Arbitrage opportunity in the gold market when gold futures price is too high

From the Trader's Desk

The one-year futures price of gold is \$500 per ounce. The spot price is \$450 per ounce and the risk-free interest rate is 7% per annum. The storage costs for gold are \$2 per ounce per year payable in arrears.

Opportunity

The futures price of gold is too high. An arbitrageur can

1. Borrow \$45,000 at the risk-free interest rate to buy 100 ounces of gold.
2. Short one gold futures contract for delivery in one year.

At the end of the year \$50,000 is received for the gold under the terms of the futures contract, \$48,263 is used to pay interest and principal on the loan, and \$200 is used to pay storage. The net gain is

$$\$50{,}000 - \$48{,}263 - \$200 = \$1{,}537$$

Suppose that instead of equation (3.15), we have

$$F_0 > (S_0 + U)e^{rT} \tag{3.17}$$

To take advantage of this opportunity, an arbitrageur can implement the following strategy:

1. Borrow an amount $S_0 + U$ at the risk-free rate and use it to purchase one unit of the commodity and to pay storage costs.
2. Short a forward contract on one unit of the commodity.

If we regard the futures contract as a forward contract, this strategy leads to a profit of $F_0 - (S_0 + U)e^{rT}$ at time T. Table 3.9 illustrates the strategy for gold. There is no problem in implementing the strategy for any commodity. However, as arbitrageurs do so, there will be a tendency for S_0 to increase and F_0 to decrease until equation (3.17) is no longer true. We conclude that equation (3.17) cannot hold for any significant length of time.

Suppose next that

$$F_0 < (S_0 + U)e^{rT} \tag{3.18}$$

In the case of investment assets such as gold and silver, we can argue that many investors hold the commodity solely for investment. When they observe the inequality in equation (3.18), they will find it profitable to:

1. Sell the commodity, save the storage costs, and invest the proceeds at the risk-free interest rate.
2. Take a long position in a forward contract.

This strategy is illustrated for gold in Table 3.10. The result is a riskless profit at maturity of $(S_0 + U)e^{rT} - F_0$ relative to the position the investors would have been in if they had held the gold or silver. It follows that equation (3.18) cannot hold for long. Because neither equation (3.17) nor (3.18) can hold for long, we must have $F_0 = (S_0 + U)e^{rT}$.

For commodities that are not to any significant extent held for investment, this argument cannot be used. Individuals and companies who keep such a commodity in

Table 3.10 Arbitrage opportunity in the gold market when gold futures price is too low

From the Trader's Desk

The one-year futures price of gold is \$470 per ounce. The spot price is \$450 per ounce and the risk-free interest rate is 7% per annum. The storage costs for gold are \$2 per ounce per year payable in arrears.

Opportunity

The futures price of gold is too low. An investor who already holds 100 ounces of gold for investment purposes can

1. Sell the gold for \$45,000.
2. Enter into one long gold futures contract for delivery in one year.

The \$45,000 is invested at the risk-free interest rate for one year and grows to \$48,263. At the end of the year, under the terms of the futures contract, 100 ounces of gold are purchased for \$47,000. The investor therefore ends up with 100 ounces of gold plus

$$\$48{,}263 - \$47{,}000 = \$1{,}263$$

in cash. If the gold is kept throughout the year, the investor ends up with 100 ounces of gold, but has to pay \$200 for storage. The futures contract therefore improves the investor's position by

$$1{,}263 + \$200 = \$1{,}463$$

inventory do so because of its consumption value—not because of its value as an investment. They are reluctant to sell the commodity and buy forward contracts, because forward contracts cannot be consumed. There is therefore nothing to stop equation (3.18) from holding. All we can assert for a consumption commodity is therefore

$$F_0 \leqslant (S_0 + U)e^{rT} \tag{3.19}$$

If storage costs are expressed as a proportion u of the spot price, the equivalent result is

$$F_0 \leqslant S_0 e^{(r+u)T} \tag{3.20}$$

Convenience Yields

We do not necessarily have equality in equations (3.19) and (3.20) because users of a consumption commodity may feel that ownership of the physical commodity provides benefits that are not obtained by holders of futures contracts. For example, an oil refiner is unlikely to regard a futures contract on crude oil in the same way as crude oil held in inventory. The crude oil in inventory can be an input to the refining process whereas a futures contract cannot be used for this purpose. In general, ownership of the physical asset enables a manufacturer to keep a production process running and perhaps profit from temporary local shortages. A futures contract does not do the same. The benefits from holding the physical asset are sometimes referred to as the *convenience yield* provided by the commodity. If the dollar amount of storage costs is known and has a present value, U, the convenience yield, y, is defined so that

$$F_0 e^{yT} = (S_0 + U)e^{rT}$$

If the storage costs per unit are a constant proportion, u, of the spot price, y is defined so that

$$F_0 e^{yT} = S_0 e^{(r+u)T}$$

or

$$F_0 = S_0 e^{(r+u-y)T} \tag{3.21}$$

The convenience yield simply measures the extent to which the left-hand side is less than the right-hand side in equation (3.19) or (3.20). For investment assets the convenience yield must be zero; otherwise, there are opportunities such as those in Table 3.10. Table 2.2 of Chapter 2 shows that the futures prices of some commodities such as Sugar-World tended to decrease as the time to maturity of the contract increased on March 15, 2001. This pattern suggests that the convenience yield, y, is greater than $r + u$ for these commodities.

The convenience yield reflects the market's expectations concerning the future availability of the commodity. The greater the possibility that shortages will occur, the higher the convenience yield. If users of the commodity have high inventories, there is very little chance of shortages in the near future and the convenience yield tends to be low. On the other hand, low inventories tend to lead to high convenience yields.

3.13 THE COST OF CARRY

The relationship between futures prices and spot prices can be summarized in terms of the *cost of carry*. This measures the storage cost plus the interest that is paid to finance the asset less the income earned on the asset. For a non-dividend-paying stock, the cost of carry is r, because there are no storage costs and no income is earned; for a stock index, it is $r - q$, because income is earned at rate q on the asset. For a currency, it is $r - r_f$; for a commodity with storage costs that are a proportion u of the price, it is $r + u$; and so on.

Define the cost of carry as c. For an investment asset, the futures price is

$$F_0 = S_0 e^{cT} \tag{3.22}$$

For a consumption asset, it is

$$F_0 = S_0 e^{(c-y)T} \tag{3.23}$$

where y is the convenience yield.

3.14 DELIVERY OPTIONS

Whereas a forward contract normally specifies that delivery is to take place on a particular day, a futures contract often allows the party with the short position to choose to deliver at any time during a certain period. (Typically the party has to give a few days' notice of its intention to deliver.) The choice introduces a complication into the determination of futures prices. Should the maturity of the futures contract be assumed to be the beginning, middle, or end of the delivery period? Even though most futures contracts are closed out prior to maturity, it is important to know when delivery would have taken place in order to calculate the theoretical futures price.

If the futures price is an increasing function of the time to maturity, it can be seen from equation (3.23) that $c > y$, so that the benefits from holding the asset (including convenience yield and net of storage costs) are less than the risk-free rate. It is usually optimal in such a case for the party with the short position to deliver as early as possible, because the interest earned on the cash received outweighs the benefits of holding the asset. As a rule, futures prices in these circumstances should be calculated on the basis that delivery will take place at the beginning of the delivery period. If futures prices are decreasing as time to maturity increases ($c < y$), the reverse is true. It is then usually optimal for the party with the short position to deliver as late as possible, and futures prices should, as a rule, be calculated on this assumption.

3.15 FUTURES PRICES AND THE EXPECTED FUTURE SPOT PRICE

One question that is often raised is whether the futures price of an asset is equal to its expected future spot price. If you have to guess what the price of an asset will be in three months, is the futures price an unbiased estimate? Chapter 2 presented the arguments of Keynes and Hicks. These authors contend that speculators will not trade a futures contract unless their expected profit is positive. By contrast, hedgers are prepared to accept a negative profit because of the risk-reduction benefits they get from a futures contract. If more speculators are long than short, the futures price will tend to be less than the expected future spot price. On average, speculators can then expect to make a gain, because the futures price converges to the spot price at maturity of the contract. Similarly, if more speculators are short than long, the futures price will tend to be greater than the expected future spot price.

Risk and Return

Another explanation of the relationship between futures prices and expected future spot prices can be obtained by considering the relationship between risk and expected return in the economy. In general, the higher the risk of an investment, the higher the expected return demanded by an investor. Readers familiar with the capital asset pricing model will know that there are two types of risk in the economy: systematic and nonsystematic. Nonsystematic risk should not be important to an investor. It can be almost completely eliminated by holding a well-diversified portfolio. An investor should not therefore require a higher expected return for bearing nonsystematic risk. Systematic risk, by contrast, cannot be diversified away. It arises from a correlation between returns from the investment and returns from the stock market as a whole. An investor generally requires a higher expected return than the risk-free interest rate for bearing positive amounts of systematic risk. Also, an investor is prepared to accept a lower expected return than the risk-free interest rate when the systematic risk in an investment is negative.

The Risk in a Futures Position

Let us consider a speculator who takes a long futures position in the hope that the spot price of the asset will be above the futures price at maturity. We suppose that the speculator puts the present value of the futures price into a risk-free investment while simultaneously taking a long futures position. We assume that the futures contract can be treated as a forward contract. The proceeds of the risk-free investment are used to buy the asset on the delivery date. The asset is then immediately sold for its market price. The cash flows to the speculator are

Time 0: $-F_0e^{-rT}$

Time T: $+S_T$

where S_T is the price of the asset at time T.

The present value of this investment is

$$-F_0e^{-rT} + E(S_T)e^{-kT}$$

where k is the discount rate appropriate for the investment (i.e., the expected return required by investors on the investment) and E denotes expected value. Assuming that all investment opportunities in securities markets have zero net present value, we have

$$-F_0e^{-rT} + E(S_T)e^{-kT} = 0$$

or

$$F_0 = E(S_T)e^{(r-k)T} \quad \textbf{(3.24)}$$

The value of k depends on the systematic risk of the investment. If S_T is uncorrelated with the level of the stock market, the investment has zero systematic risk. In this case $k = r$, and equation (3.24) shows that $F_0 = E(S_T)$. If S_T is positively correlated with the stock market as a whole, the investment has positive systematic risk. In this case $k > r$, and equation (3.24) shows that $F_0 < E(S_T)$. Finally, if S_T is negatively correlated with the stock market, the investment has negative systematic risk. In this case $k < r$, and equation (3.24) shows that $F_0 > E(S_T)$.

Empirical Evidence

If $F_0 = E(S_T)$, the futures price will drift up or down only if the market changes its views about the expected future spot price. Over a long period of time, we can reasonably assume that the market revises its expectations about future spot prices upward as often as it does so downward. It follows that when $F_0 = E(S_T)$, the average profit from holding futures contracts over a long period of time should be zero. The $F_0 < E(S_T)$ situation corresponds to the positive systematic risk situation. Because the futures price and the spot price must be equal at maturity of the futures contract, a futures price should on average drift up, and a trader should over a long period of time make positive profits from consistently holding long futures positions. Similarly, the $F_0 > E(S_T)$ situation implies that a trader should over a long period of time make positive profits from consistently holding short futures positions.

How do futures prices behave in practice? Some of the empirical work that has been carried out is listed at the end of this chapter. The results are mixed. Houthakker's study looked at futures prices for wheat, cotton, and corn from 1937 to 1957. It showed that significant profits could be earned by taking long futures positions. This suggests that an investment in corn has positive systematic risk and $F_0 < E(S_T)$. Telser's study contradicted the findings of Houthakker. His data covered the period from 1926 to 1950 for cotton and from 1927 to 1954 for wheat and gave rise to no significant profits for traders taking either long or short positions. To quote from Telser: "The futures data offer no evidence to contradict the simple...hypothesis that the futures price is an unbiased estimate of the expected future spot price." Gray's study looked at corn futures prices during the period 1921 to 1959 and resulted in similar findings to those of Telser. Dusak's study used data on corn, wheat, and soybeans from 1952 to 1967 and took a different approach. It attempted to estimate the systematic risk of an investment in these commodities by calculating the correlation of movements in the commodity prices with movements in the S&P 500. The results suggest that there is no systematic risk and lend support to the $F_0 = E(S_T)$ hypothesis. However, more recent work by Chang using the same commodities and more advanced statistical techniques supports the $F_0 < E(S_T)$ hypothesis.

3.16 SUMMARY

For most purposes, the futures price of a contract with a certain delivery date can be considered to be the same as the forward price for a contract with the same delivery date. It can be shown that in theory the two should be exactly the same when interest rates are perfectly predictable.

For the purposes of understanding futures (or forward) prices, it is convenient to divide futures contracts into two categories: those in which the underlying asset is held for investment by a significant number of investors and those in which the underlying asset is held primarily for consumption purposes.

In the case of investment assets, we have considered three different situations:

1. The asset provides no income.
2. The asset provides a known dollar income.
3. The asset provides a known yield.

The results are summarized in Table 3.11. They enable futures prices to be obtained for contracts on stock indices, currencies, gold, and silver. Storage costs can be regarded as negative income.

In the case of consumption assets, it is not possible to obtain the futures price as a function of the spot price and other observable variables. Here the parameter known as the asset's convenience yield becomes important. It measures the extent to which users of the commodity feel that ownership of the physical asset provides benefits that are not obtained by the holders of the futures contract. These benefits may include the ability to profit from temporary local shortages or the ability to keep a production process running. We can obtain an upper bound for the futures price of consumption assets using arbitrage arguments, but we cannot nail down an equality relationship between futures and spot prices.

The concept of cost of carry is sometimes useful. The cost of carry is the storage cost of the underlying asset plus the cost of financing it minus the income received from it.

Table 3.11 Summary of results for a contract with time to maturity T on an investment asset with price S_0 when the risk-free interest rate for a T-year period is r

Asset	Forward/futures price	Value of long forward contract with delivery price K
Provides no income	S_0e^{rT}	$S_0 - Ke^{-rT}$
Provides known income with present value, I	$(S_0 - I)e^{rT}$	$S_0 - I - Ke^{-rT}$
Provides known yield, q	$S_0e^{(r-q)T}$	$S_0e^{-qT} - Ke^{-rT}$

In the case of investment assets, the futures price is greater than the spot price by an amount reflecting the cost of carry. In the case of consumption assets, the futures price is greater than the spot price by an amount reflecting the cost of carry net of the convenience yield.

If we assume the capital asset pricing model is true, the relationship between the futures price and the expected future spot price depends on whether the return on the asset is positively or negatively correlated with the return on the stock market. Positive correlation will tend to lead to a futures price lower than the expected future spot price. Negative correlation will tend to lead to a futures price higher than the expected future spot price. Only when the correlation is zero will the theoretical futures price be equal to the expected future spot price.

Suggestions for Further Reading

On empirical research concerning forward and futures prices

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Houthakker, H. S. "Can Speculators Forecast Prices?" *Review of Economics and Statistics* 39 (1957): 143–51.

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On the theoretical relationship between forward and futures prices

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Jarrow, R. A., and G. S. Oldfield. "Forward Contracts and Futures Contracts." *Journal of Financial Economics* 9 (December 1981): 373–82.

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Quiz (Answers at End of Book)

3.1. A bank quotes you an interest rate of 14% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

3.2. Explain what happens when an investor shorts a certain share.

3.3. Suppose that you enter into a six-month forward contract on a non-dividend-paying stock when the stock price is $30 and the risk-free interest rate (with continuous compounding) is 12% per annum. What is the forward price?

3.4. A stock index currently stands at 350. The risk-free interest rate is 8% per annum (with continuous compounding) and the dividend yield on the index is 4% per annum. What should the futures price for a four-month contract be?

3.5. Explain carefully why the futures price of gold can be calculated from its spot price and other observable variables whereas the futures price of copper cannot.

3.6. Explain carefully the meaning of the terms *convenience yield* and *cost of carry*. What is the relationship between futures price, spot price, convenience yield, and cost of carry?

3.7. Is the futures price of a stock index greater than or less than the expected future value of the index? Explain your answer.

Questions and Problems (Answers in Solutions Manual)

3.8. An investor receives $1,100 in one year in return for an investment of $1,000 now. Calculate the percentage return per annum with
a. Annual compounding
b. Semiannual compounding
c. Monthly compounding
d. Continuous compounding

3.9. What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

3.10. A deposit account pays 12% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a $10,000 deposit?

3.11. A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is $40 and the risk-free rate of interest is 10% per annum with continuous compounding.
a. What are the forward price and the initial value of the forward contract?
b. Six months later, the price of the stock is $45 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?

3.12. The risk-free rate of interest is 7% per annum with continuous compounding, and the dividend yield on a stock index is 3.2% per annum. The current value of the index is 150. What is the six-month futures price?

3.13. Assume that the risk-free interest rate is 9% per annum with continuous compounding and that the dividend yield on a stock index varies throughout the year. In February, May, August, and November, dividends are paid at a rate of 5% per annum. In other months, dividends are paid at a rate of 2% per annum. Suppose that the value of the index on July 31, 2001, is 300. What is the futures price for a contract deliverable on December 31, 2001?

3.14. Suppose that the risk-free interest rate is 10% per annum with continuous compounding and that the dividend yield on a stock index is 4% per annum. The index is standing at 400, and the futures price for a contract deliverable in four months is 405. What arbitrage opportunities does this create?

3.15. Estimate the difference between short-term interest rates in Mexico and the United States on March 15, 2001, from the information in Table 3.8.

3.16. The two-month interest rates in Switzerland and the United States are 3% and 8% per annum, respectively, with continuous compounding. The spot price of the Swiss franc is $0.6500. The futures price for a contract deliverable in two months is $0.6600. What arbitrage opportunities does this create?

3.17. The current price of silver is $9 per ounce. The storage costs are $0.24 per ounce per year payable quarterly in advance. Assuming that interest rates are 10% per annum for all maturities, calculate the futures price of silver for delivery in nine months.

3.18. Suppose that F_1 and F_2 are two futures contracts on the same commodity with times to maturity t_1 and t_2, where $t_2 > t_1$. Prove that

$$F_2 \leqslant F_1 e^{r(t_2 - t_1)}$$

where r is the interest rate (assumed constant) and there are no storage costs. For the purposes of this problem, assume that a futures contract is the same as a forward contract.

3.19. When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the marking-to-market process does leave the company exposed to some risk. Explain the nature of this risk. In particular, consider whether the company is better off using a futures contract or a forward contract when

a. The value of the foreign currency falls rapidly during the life of the contract
b. The value of the foreign currency rises rapidly during the life of the contract
c. The value of the foreign currency first rises and then falls back to its initial value
d. The value of the foreign currency first falls and then rises back to its initial value

Assume that the forward price equals the futures price.

3.20. It is sometimes argued that a forward exchange rate is an unbiased predictor of future exchange rates. Under what circumstances is this so?

3.21. Show that the growth rate in an index futures price equals the excess return of the index over the risk-free rate. Assume that the risk-free interest rate and the dividend yield are constant.

3.22. Show that equation (3.7) is true by considering an investment in the asset combined with a short position in a futures contract. Assume that all income from the asset is reinvested in the asset. Use an argument similar to that in footnotes 2 and 3 and explain in detail what an arbitrageur would do if equation (3.7) did not hold.

Assignment Questions

3.23. A stock is expected to pay a dividend of $1 per share in two months and in five months. The stock price is $50, and the risk-free rate of interest is 8% per annum with continuous compounding for all maturities. An investor has just taken a short position in a six-month forward contract on the stock.

a. What are the forward price and the initial value of the forward contract?

b. Three months later, the price of the stock is $48 and the risk-free rate of interest is still 8% per annum. What are the forward price and the value of the short position in the forward contract?

3.24. A bank offers a corporate client a choice between borrowing cash at 11% per annum and borrowing gold at 2% per annum. (If gold is borrowed, interest must be repaid in gold. Thus, 100 ounces borrowed today would require 102 ounces to be repaid in one year.) The risk-free interest rate is 9.25% per annum, and storage costs are 0.5% per annum. Discuss whether the rate of interest on the gold loan is too high or too low in relation to the rate of interest on the cash loan. The interest rates on the two loans are expressed with annual compounding. The risk-free interest rate and storage costs are expressed with continuous compounding.

3.25. A company that is uncertain about the exact date when it will pay or receive a foreign currency may try to negotiate with its bank a forward contract that specifies a period during which delivery can be made. The company wants to reserve the right to choose the exact delivery date to fit in with its own cash flows. Put yourself in the position of the bank. How would you price the product that the company wants?

3.26. A foreign exchange trader working for a bank enters into a long forward contract to buy 1 million pounds sterling at an exchange rate of 1.6000 in three months. At the same time, another trader on the next desk takes a long position in 16 three-month futures contracts on sterling. The futures price is 1.6000, and each contract is on 62,500 pounds. Within minutes of the trades being executed the forward and the futures prices both increase to 1.6040. Both traders immediately claim a profit of $4,000. The bank's systems show that the futures trader has made a $4,000 profit, but the forward trader has made a profit of only $3,900. The forward trader immediately picks up the phone to complain to the systems department. Explain what is going on here. Why are the profits different?

3.27. A trader owns gold as part of a long-term investment portfolio. The trader can buy gold for $250 per ounce and sell gold for $249 per ounce. The trader can borrow funds at 6% per year and invest funds at 5.5% per year. (Both interest rates are expressed with annual compounding.) For what range of one-year forward prices of gold does the trader have no arbitrage opportunities? Assume there is no bid–offer spread for forward prices.

APPENDIX

Proof That Forward and Futures Prices Are Equal When Interest Rates Are Constant

This appendix demonstrates that forward and futures prices are equal when interest rates are constant. Suppose that a futures contract lasts for n days and that F_i is the futures price at the end of day i $(0 < i < n)$. Define δ as the risk-free rate per day (assumed constant). Consider the following strategy.[6]

1. Take a long futures position of e^{δ} at the end of day 0 (i.e., at the beginning of the contract).
2. Increase long position to $e^{2\delta}$ at the end of day 1.
3. Increase long position to $e^{3\delta}$ at the end of day 2.

And so on.

This strategy is summarized in Table 3.12. By the beginning of day i, the investor has a long position of $e^{\delta i}$. The profit (possibly negative) from the position on day i is

$$(F_i - F_{i-1})e^{\delta i}$$

Assume that the profit is compounded at the risk-free rate until the end of day n. Its value at the end of day n is

$$(F_i - F_{i-1})e^{\delta i}e^{(n-i)\delta} = (F_i - F_{i-1})e^{n\delta}$$

The value at the end of day n of the entire investment strategy is therefore

$$\sum_{i=1}^{n}(F_i - F_{i-1})e^{n\delta}$$

This is

$$[(F_n - F_{n-1}) + (F_{n-1} - F_{n-2}) + \cdots + (F_1 - F_0)]e^{n\delta} = (F_n - F_0)e^{n\delta}$$

Because F_n is the same as the terminal asset spot price, S_T, the terminal value of the investment strategy can be written

$$(S_T - F_0)e^{n\delta}$$

An investment of F_0 in a risk-free bond combined with the strategy just given yields

$$F_0e^{n\delta} + (S_T - F_0)e^{n\delta} = S_Te^{n\delta}$$

at time T. No investment is required for all the long futures positions described. It follows that an amount F_0 can be invested to give an amount $S_Te^{n\delta}$ at time T.

Suppose next that the forward price at the end of day 0 is G_0. Investing G_0 in a riskless bond and taking a long forward position of $e^{n\delta}$ forward contracts also guarantees an amount $S_Te^{n\delta}$ at time T. Thus, there are two investment strategies—

[6] This strategy was proposed by J. C. Cox, J. E. Ingersoll, and S. A. Ross, "The Relation between Forward Prices and Futures Prices," *Journal of Financial Economics* 9 (December 1981): 321–46.

Table 3.12 The investment strategy to show that futures and forward prices are equal

Day	0	1	2	...	$n-1$	n
Futures price	F_0	F_1	F_2	...	F_{n-1}	F_n
Futures position	e^{δ}	$e^{2\delta}$	$e^{3\delta}$	...	$e^{n\delta}$	0
Gain/loss	0	$(F_1 - F_0)e^{\delta}$	$(F_2 - F_1)e^{2\delta}$	...	...	$(F_n - F_{n-1})e^{n\delta}$
Gain/loss compounded to day n	0	$(F_1 - F_0)e^{n\delta}$	$(F_2 - F_1)e^{n\delta}$	...	...	$(F_n - F_{n-1})e^{n\delta}$

one requiring an initial outlay of F_0 and the other requiring an initial outlay of G_0—both of which yield $S_T e^{n\delta}$ at time T. It follows that in the absence of arbitrage opportunities

$$F_0 = G_0$$

In other words, the futures price and the forward price are identical. Note that in this proof there is nothing special about the time period of one day. The futures price based on a contract with weekly settlements is also the same as the forward price when corresponding assumptions are made.