

# 5

CHAPTER

# Interest Rate Markets

In this chapter we cover many different aspects of interest rate markets. We explain zero rates, par yields, forward rates, and the relationships between them. We cover day count conventions and the way in which the prices of Treasury bonds, corporate bonds, and Treasury bills are quoted in the United States. We show how forward rate agreements can be valued. We discuss the duration measure and explain how it can be used to quantify a company's exposure to interest rates. We also consider interest rate futures markets. We describe in some detail the popular Treasury bond futures and Eurodollar futures contracts that trade in the United States, and we examine how they can be used for duration-based hedging.

## 5.1 TYPES OF RATES

For any given currency, many different types of interest rates are regularly quoted. These include mortgage rates, deposit rates, prime borrowing rates, and so on. The interest rate applicable in a situation depends on the credit risk. The higher the credit risk, the higher the interest rate. In this section, we introduce three interest rates that are particularly important in options and futures markets.

### Treasury Rates

Treasury rates are the interest rates applicable to borrowing by a government in its own currency. For example, Italian Treasury rates are the rates at which the Italian government can borrow in lira; Japanese Treasury rates are the rates at which the Japanese government can borrow in yen; and so on. It is usually assumed that there is no chance that a government will default on an obligation denominated in its own currency.<sup>1</sup> For this reason, Treasury rates are often termed risk-free rates.

### LIBOR Rates

LIBOR is short for the *London Interbank Offer Rate*. It is the rate at which large international banks fund much of their activities. Specifically, it is the rate at which one large international bank is willing to lend money to another large international bank.

<sup>1</sup> The reason for this is that the government can always meet its obligation by printing more money.

LIBOR rates are determined in trading between banks and change as economic conditions change. LIBOR rates are generally higher than the corresponding Treasury rates because they are not risk-free rates. There is always some chance (albeit small) that the bank borrowing the money will default.

As mentioned in Chapter 3, banks and other large financial institutions tend to use the LIBOR rate rather than the Treasury rate as the “risk-free rate” when they evaluate derivatives transactions. The reason is that financial institutions invest surplus funds in the LIBOR market and borrow to meet their short-term funding requirements in this market. They regard LIBOR as their opportunity cost of capital.

### Repo Rate

Sometimes an investment dealer funds its trading activities with a *repo* or *repurchase agreement*. This is a contract where an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price. The company is providing a loan to the investment dealer. The difference between the price at which the securities are sold and the price at which they are repurchased is the interest it earns. The interest rate is referred to as the *repo rate*. If structured carefully, the loan involves very little credit risk. If the original owner of the securities does not keep to its side of the agreement, the lending company simply keeps the securities. If the lending company does not keep to its side of the agreement, the original owner of the securities keeps the cash.

The most common type of repo is an *overnight repo*, in which the agreement is renegotiated each day. However, longer-term arrangements, known as *term repos*, are sometimes used.

## 5.2 ZERO RATES

The *n*-year *zero rate* (short for zero-coupon rate) is the rate of interest earned on an investment that starts today and lasts for *n* years. All the interest and principal is realized at the end of *n* years. There are no intermediate payments. The *n*-year zero rate is sometimes also referred to as the *n*-year *spot rate*. Suppose the five-year Treasury zero rate with continuous compounding is quoted as 5% per annum. This means that \$100, if invested at the risk-free rate for five years, would grow to

$$100 \times e^{0.05 \times 5} = 128.40$$

Many of the interest rates we observe directly in the market are not pure zero rates. Consider a five-year government bond that provides a 6% coupon. The price of this bond does not exactly determine the five-year Treasury zero rate because some of the return on the bond is realized in the form of coupons prior to the end of year five. Later in this chapter we will discuss how we can determine Treasury zero rates and LIBOR zero rates from the prices of traded instruments.

## 5.3 BOND PRICING

Most bonds provide coupons periodically. The owner also receives the principal or face value of the bond at maturity. The theoretical price of a bond can be calculated as the

**Table 5.1** Treasury zero rates

Maturity (years)	Zero rate (%) (cont. comp.)
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

present value of all the cash flows that will be received by the owner of the bond using the appropriate zero rates as discount rates. Consider the situation where Treasury zero rates, measured with continuous compounding, are as in Table 5.1. (We explain later how these can be calculated.) Suppose that a two-year Treasury bond with a principal of \$100 provides coupons at the rate of 6% per annum semiannually. To calculate the present value of the first coupon of \$3, we discount it at 5.0% for six months; to calculate the present value of the second coupon of \$3, we discount it at 5.8% for one year; and so on. The theoretical price of the bond is therefore

$$3e^{-0.05 \times 0.5} + 3e^{-0.058 \times 1.0} + 3e^{-0.064 \times 1.5} + 103e^{-0.068 \times 2.0} = 98.39$$

or \$98.39.

### Bond Yield

The yield on a coupon-bearing bond is the discount rate that equates the cash flows on the bond to its market value. Suppose that the theoretical price of the bond we have been considering, \$98.39, is also its market value (that is, the market's price of the bond is in exact agreement with the data in Table 5.1). If  $y$  is the yield on the bond, expressed with continuous compounding, we must have

$$3e^{-y \times 0.5} + 3e^{-y \times 1.0} + 3e^{-y \times 1.5} + 103e^{-y \times 2.0} = 98.39$$

This equation can be solved using an iterative ("trial and error") procedure to give  $y = 6.76\%$ .

### Par Yield

The *par yield* for a certain maturity is the coupon rate that causes the bond price to equal its face value. Usually the bond is assumed to provide semiannual coupons. Suppose that the coupon on a two-year bond in our example is  $c$  per annum (or  $c/2$  per six months). Using the zero rates in Table 5.1, the value of the bond is equal to its face value of 100 when

$$\frac{c}{2}e^{-0.05 \times 0.5} + \frac{c}{2}e^{-0.058 \times 1.0} + \frac{c}{2}e^{-0.064 \times 1.5} + \left(100 + \frac{c}{2}\right)e^{-0.068 \times 2.0} = 100$$

This equation can be solved in a straightforward way to give  $c = 6.87$ . The two-year par yield is therefore 6.87% per annum with semiannual compounding (or 6.75% with continuous compounding).

More generally, if  $d$  is the present value of \$1 received at the maturity of the bond,

$A$  is the value of an annuity that pays one dollar on each coupon payment date, and  $m$  is the number of coupon payments per year, the par yield  $c$  must satisfy

$$100 = A \frac{c}{m} + 100d$$

so that

$$c = \frac{(100 - 100d)m}{A}$$

In our example,  $m = 2$ ,  $d = e^{-0.068 \times 2} = 0.87284$ , and

$$A = e^{-0.05 \times 0.5} + e^{-0.058 \times 1.0} + e^{-0.064 \times 1.5} + e^{-0.068 \times 2.0} = 3.70027.$$

The formula confirms that the par yield is 6.87% per annum with semiannual compounding.

## 5.4 DETERMINING TREASURY ZERO RATES

We now discuss how Treasury zero rates can be calculated from the prices of instruments that trade. One approach is known as the *bootstrap method*. To illustrate the nature of the method, consider the data in Table 5.2 on the prices of five bonds. Because the first three bonds pay no coupons, the zero rates corresponding to the maturities of these bonds can be easily calculated. The three-month bond provides a return of 2.5 in three months on an initial investment of 97.5. With quarterly compounding, the three-month zero rate is  $(4 \times 2.5)/97.5 = 10.256\%$  per annum. Equation (3.3) shows that, when the rate is expressed with continuous compounding, it becomes

$$4 \ln \left( 1 + \frac{0.10256}{4} \right) = 0.10127$$

or 10.127% per annum. The six-month bond provides a return of 5.1 in six months on an initial investment of 94.9. With semiannual compounding the six-month rate is  $(2 \times 5.1)/94.9 = 10.748\%$  per annum. Equation (3.3) shows that, when the rate is expressed with continuous compounding, it becomes

$$2 \ln \left( 1 + \frac{0.10748}{2} \right) = 0.10469$$

**Table 5.2** Data for bootstrap method

Bond principal (\$)	Time to maturity (years)	Annual coupon (\$)*	Bond price (\$)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

\* Half the stated coupon is assumed to be paid every six months.

**Table 5.3** Continuously compounded zero rates determined from data in Table 5.2

Maturity (years)	Zero rate (%) (cont. comp.)
0.25	10.127
0.50	10.469
1.00	10.536
1.50	10.681
2.00	10.808

or 10.469% per annum. Similarly, the one-year rate with continuous compounding is

$$\ln\left(1 + \frac{10}{90}\right) = 0.10536$$

or 10.536% per annum.

The fourth bond lasts 1.5 years. The payments are as follows:

6 months: \$4

1 year: \$4

1.5 years: \$104

From our earlier calculations, we know that the discount rate for the payment at the end of six months is 10.469%, and the discount rate for the payment at the end of one year is 10.536%. We also know that the bond's price, \$96, must equal the present value of all the payments received by the bondholder. Suppose the 1.5-year zero rate is denoted by  $R$ . It follows that

$$4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96$$

This reduces to

$$e^{-1.5R} = 0.85196$$

or

$$R = -\frac{\ln(0.85196)}{1.5} = 0.10681$$

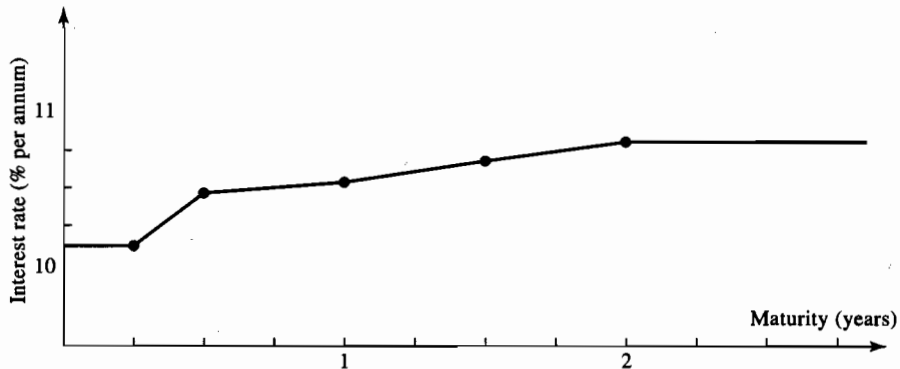
The 1.5-year zero rate is therefore 10.681%. This is the only zero rate that is consistent with the six-month rate, one-year rate, and the data in Table 5.3.

The two-year zero rate can be calculated similarly from the six-month, one-year, and 1.5-year zero rates and the information on the last bond in Table 5.2. If  $R$  is the two-year zero rate,

$$6e^{-0.10469 \times 0.5} + 6e^{-0.10536 \times 1.0} + 6e^{-0.10681 \times 1.5} + 106e^{-R \times 2.0} = 101.6$$

This gives  $R = 0.10808$ , or 10.808%.

The rates we have calculated are summarized in Table 5.3. A chart showing the zero rate as a function of maturity is known as the *zero curve*. A common assumption is that the zero curve is linear between the points determined using the bootstrap method. (This means that the 1.25-year zero rate is  $0.5 \times 10.536 + 0.5 \times 10.681 = 10.6085\%$  in our example.) It is also usually assumed that the zero curve is horizontal prior to the first point and horizontal beyond the last point. Figure 5.1 shows the zero curve for our



**Figure 5.1** Zero rates given by the bootstrap method

data. By using longer maturity bonds, the zero curve would be more accurately determined beyond two years.

In practice, we do not usually have bonds with maturities equal to exactly 1.5 years, 2 years, 2.5 years, and so on. The approach often used by analysts is to interpolate between the bond price data before it is used to calculate the zero curve. For example, if they know that a 2.3-year bond with a coupon of 6% sells for 98 and a 2.7-year bond with a coupon of 6.5% sells for 99, they might assume that a 2.5-year bond with a coupon of 6.25% would sell for 98.5.

## 5.5 FORWARD RATES

Forward interest rates are the rates of interest implied by current zero rates for periods of time in the future. To illustrate how they are calculated, we suppose that the zero rates are as shown in the second column of Table 5.4. The rates are assumed to be continuously compounded. Thus, the 10% per annum rate for one year means that, in return for an investment of \$100 today, the investor receives  $100e^{0.1} = \$110.52$  in one year; the 10.5% per annum rate for two years means that, in return for an investment of \$100 today, the investor receives  $100e^{0.105 \times 2} = \$123.37$  in two years; and so on.

The forward interest rate in Table 5.4 for year 2 is 11% per annum. This is the rate of interest that is implied by the zero rates for the period of time between the end of the first year and the end of the second year. It can be calculated from the one-year zero interest rate of 10% per annum and the two-year zero interest rate of 10.5% per annum.

**Table 5.4** Calculation of forward rates

Year (n)	Zero rate for an n-year investment (% per annum)	Forward rate for nth year (% per annum)
1	10.0	
2	10.5	11.0
3	10.8	11.4
4	11.0	11.6
5	11.1	11.5

It is the rate of interest for year 2 that, when combined with 10% per annum for year 1, gives 10.5% overall for the two years. To show that the correct answer is 11% per annum, suppose that \$100 is invested. A rate of 10% for the first year and 11% for the second year yields

$$100e^{0.1}e^{0.11} = \$123.37$$

at the end of the second year. A rate of 10.5% per annum for two years yields

$$100e^{0.105 \times 2}$$

which is also \$123.37. This example illustrates the general result that when interest rates are continuously compounded and rates in successive time periods are combined, the overall equivalent rate is simply the average rate during the whole period. (In our example 10% for the first year and 11% for the second year average to 10.5% for the two years.) The result is only approximately true when the rates are not continuously compounded.

The forward rate for the year 3 is the rate of interest that is implied by a 10.5% per annum two-year zero rate and a 10.8% per annum three-year zero rate. It is 11.4% per annum. The reason is that an investment for two years at 10.5% per annum combined with an investment for one year at 11.4% per annum gives an overall average return for the three years of 10.8% per annum. The other forward rates can be calculated similarly and are shown in the third column of the table. In general, if  $R_1$  and  $R_2$  are the zero rates for maturities  $T_1$  and  $T_2$ , respectively, and  $R_F$  is the forward interest rate for the period of time between  $T_1$  and  $T_2$ , then

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1} \quad (5.1)$$

To illustrate this formula, consider the calculation of the year 4 forward rate from the data in Table 5.4:  $T_1 = 3$ ,  $T_2 = 4$ ,  $R_1 = 0.108$ , and  $R_2 = 0.11$ , and the formula gives  $R_F = 0.116$ .

Assuming that the zero rates for borrowing and investing are the same (which is not too unreasonable for a large financial institution), an investor can lock in the forward rate for a future time period. Suppose for example that the zero rates are as in Table 5.4. If an investor borrows \$100 at 10% for one year and then invests the money at 10.5% for two years, the result is a cash outflow of  $100e^{0.1} = \$110.52$  at the end of year one and an inflow of  $100e^{0.105 \times 2} = \$123.37$  at the end of year two. Because  $123.37 = 110.52e^{0.11}$ , a return equal to the forward rate (11%) is earned on \$110.52 during the second year. Suppose next that the investor borrows \$100 for four years at 11% and invests it for three years at 10.8%. The result is a cash inflow of  $100e^{0.108 \times 3} = \$138.26$  at the end of the third year and a cash outflow of  $100e^{0.11 \times 4} = \$155.27$  at the end of the fourth year. Because  $155.27 = 138.26e^{0.116}$ , money is being borrowed for the fourth year at the forward rate of 11.6%.

Equation (5.1) can be written

$$R_F = R_2 + (R_2 - R_1) \frac{T_1}{T_2 - T_1} \quad (5.2)$$

This shows that if the zero curve is upward sloping between  $T_1$  and  $T_2$ , so that  $R_2 > R_1$ , then  $R_F > R_2$ . Similarly if the zero curve is downward sloping with  $R_2 < R_1$ , then  $R_F < R_2$ .

## 5.6 FORWARD RATE AGREEMENTS

A *forward rate agreement* (FRA) is an over-the-counter agreement that a certain interest rate will apply to a certain principal during a specified future period of time. In this section we examine how forward rate agreements can be valued in terms of forward rates.

Consider a forward rate agreement in which it is agreed that a financial institution will earn an interest rate  $R_K$  for the period of time between  $T_1$  and  $T_2$  on a principal of  $L$ . Define:

$R_F$ : The forward LIBOR interest rate for the period between times  $T_1$  and  $T_2$

$R$ : The actual LIBOR interest rate observed at time  $T_1$  for a maturity  $T_2$

We will depart from our usual assumption of continuous compounding and assume that the rates  $R_K$ ,  $R_F$ , and  $R$  are all measured with a compounding frequency reflecting their maturity. This means that if  $T_2 - T_1 = 0.5$ , they are expressed with semiannual compounding; if  $T_2 - T_1 = 0.25$ , they are expressed with quarterly compounding; and so on. The forward rate agreement is an agreement to the following two cash flows:<sup>2</sup>

Time  $T_1$ :  $-L$

Time  $T_2$ :  $+L[1 + R_K(T_2 - T_1)]$

To value the FRA we first note that it is always worth zero when  $R_K = R_F$ .<sup>3</sup> This is because, as noted in the previous section, a large financial institution can at no cost lock in the forward rate for a future time period. For example, it can ensure that it earns the forward rate for the time period between years 2 and 3 by borrowing for a certain amount of money for two years and investing it for three years. Similarly, it can ensure that it pays the forward rate for the time period between years 2 and 3 by borrowing for a certain amount of money for three years and investing it for two years.

We can now use an argument analogous to that used in Section 3.8 to calculate the value of the FRA for values of  $R_K$  other than  $R_F$ . Compare two FRAs. The first promises that the forward rate  $R_F$  will be earned on a principal of  $L$  between times  $T_1$  and  $T_2$ ; the second promises that  $R_K$  will be earned on the same principal between the same two dates. The two contracts are the same except for the interest payments received at time  $T_2$ . The excess of the value of the second contract over the first is, therefore, the present value of the difference between these interest payments, or

$$L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

where  $R_2$  is the continuously compounded zero rate for a maturity  $T_2$ .<sup>4</sup> Because the value of the FRA promising  $R_F$  is zero, the value of the FRA promising  $R_K$  is

$$V = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2} \quad (5.3)$$

<sup>2</sup> In practice an FRA such as the one considered is usually settled in cash at time  $T_1$ . The cash settlement is the present value of the cash flows or

$$L \frac{1 + R_K(T_2 - T_1)}{1 + R(T_2 - T_1)} - L$$

<sup>3</sup> It is usually the case that  $R_K$  is set equal to  $R_F$  when the FRA is first initiated.

<sup>4</sup> Note that  $R_K$ ,  $R$ , and  $R_F$  are expressed with a compounding frequency corresponding to  $T_2 - T_1$ , whereas  $R_2$  is expressed with continuous compounding.

When an FRA specifies that the interest rate  $R_K$  will be paid rather than received, its value is similarly given by

$$V = L(R_F - R_K)(T_2 - T_1)e^{-R_1 T_2} \quad (5.4)$$

### Example

Suppose that LIBOR rates are as in Table 5.4. Consider an FRA where we will receive a rate of 12%, measured with annual compounding, on a principal of \$1 million between the end of year 1 and the end of year 2. In this case, the forward rate is 11% with continuous compounding or 11.6278% with annual compounding. From equation (5.3) it follows that the value of the FRA is

$$1,000,000(0.12 - 0.116278)e^{-0.105 \times 2} = \$3,017$$

### Alternative Characterization of FRAs

Consider again an FRA that guarantees that a rate  $R_K$  will be earned between times  $T_1$  and  $T_2$ . The principal  $L$  can be borrowed at rate  $R$  at time  $T_1$  and repaid at time  $T_2$ . When this transaction is combined with the FRA, we see that the FRA is equivalent to the following cash flows:

Time  $T_1$ : 0

Time  $T_2$ :  $LR_K(T_2 - T_1)$

Time  $T_2$ :  $-LR(T_2 - T_1)$

In other words, the FRA is equivalent to an agreement where at time  $T_2$  interest at the predetermined rate,  $R_K$ , is received and interest at the market rate,  $R$ , is paid.

Combining the two cash flows at time  $T_2$ , we see that the FRA is equivalent to a single cash flow of

$$L(R_K - R)(T_2 - T_1)$$

at time  $T_2$ . Comparing this with equation (5.3), we see that the FRA can be valued by assuming that  $R = R_F$  and discounting the resultant cash flows at the risk-free rate. We have therefore shown that:

1. An FRA is equivalent to an agreement where interest at a predetermined rate,  $R_K$ , is exchanged for interest at the market rate,  $R$ .
2. An FRA can be valued by assuming that forward interest rate is certain to be realized.

These two results will be useful when we consider interest rate swaps in the next chapter.

## 5.7 THEORIES OF THE TERM STRUCTURE

It is natural to ask what determines the shape of the zero curve. Why is it sometimes downward sloping, sometimes upward sloping, and sometimes partly upward sloping and partly downward sloping? A number of different theories have been proposed. The simplest is *expectations theory*, which conjectures that long-term interest rates should reflect expected future short-term interest rates. More precisely, it argues that a forward interest rate corresponding to a certain future period is equal to the expected future zero

interest rate for that period. Another idea, *segmentation theory*, conjectures that there need be no relationship between short-, medium-, and long-term interest rates. Under the theory, a major investor such as a large pension fund invests in bonds of a certain maturity and does not readily switch from one maturity to another. The short-term interest rate is determined by supply and demand in the short-term bond market; the medium-term interest rate is determined by supply and demand in the medium-term bond market; and so on.

The theory that is in some ways most appealing is *liquidity preference theory*, which argues that forward rates should always be higher than expected future zero rates. The basic assumption underlying the theory is that investors prefer to preserve their liquidity and invest funds for short periods of time. Borrowers, on the other hand, usually prefer to borrow at fixed rates for long periods of time. If the interest rates offered by banks and other financial intermediaries corresponded to expectations theory, long-term interest rates would equal the average of expected future short-term interest rates. In the absence of any incentive to do otherwise, investors would tend to deposit their funds for short time periods, and borrowers would tend to choose to borrow for long time periods. Financial intermediaries would then find themselves financing substantial amounts of long-term fixed-rate loans with short-term deposits. Excessive interest rate risk would result. In practice, in order to match depositors with borrowers and avoid interest rate risk, financial intermediaries raise long-term interest rates relative to expected future short-term interest rates. This strategy reduces the demand for long-term fixed-rate borrowing and encourages investors to deposit their funds for long terms.

Liquidity preference theory leads to a situation in which forward rates are greater than expected future zero rates. It is also consistent with the empirical result that yield curves tend to be upward sloping more often than they are downward sloping.

## 5.8 DAY COUNT CONVENTIONS

We now examine the day count conventions that are used when interest rates are quoted. This is a quite separate issue from the compounding frequency used to measure interest rates. (For a discussion of the latter issue, see Section 3.3.) The day count defines the way in which interest accrues over time. Generally, we know the interest earned over some reference period (e.g., the time between coupon payments), and we are interested in calculating the interest earned over some other period.

The day count convention is usually expressed as  $X/Y$ . When we are calculating the interest earned between two dates,  $X$  defines the way in which the number of days between the two dates is calculated, and  $Y$  defines the way in which the total number of days in the reference period is measured. The interest earned between the two dates is

$$\frac{\text{Number of days between dates}}{\text{Number of days in reference period}} \times \text{Interest earned in reference period}$$

Three day count conventions that are commonly used in the United States are

1. Actual/actual (in period).
2. 30/360.
3. Actual/360.

Actual/actual (in period) is used for U.S. Treasury bonds, 30/360 is used for U.S.

corporate and municipal bonds, and actual/360 is used for U.S. Treasury bills and other money market instruments.

The use of actual/actual (in period) for Treasury bonds indicates that the interest earned between two dates is based on the ratio of the actual days elapsed to the actual number of days in the period between coupon payments. Suppose that the bond principal is \$100, coupon payment dates are March 1 and September 1, and the coupon rate is 8%. We wish to calculate the interest earned between March 1 and July 3. The reference period is from March 1 to September 1. There are 184 (actual) days in this period, and interest of \$4 is earned during the period. There are 124 (actual) days between March 1 and July 3. The interest earned between March 1 and July 3 is therefore

$$\frac{124}{184} \times 4 = 2.6957$$

The use of 30/360 for corporate and municipal bonds indicates that we assume 30 days per month and 360 days per year when carrying out calculations. With 30/360, the total number of days between March 1 and September 1 is 180. The total number of days between March 1 and July 3 is  $(4 \times 30) + 2 = 122$ . In a corporate bond with the same terms as the Treasury bond just considered, the interest earned between March 1 and July 3 would therefore be

$$\frac{122}{180} \times 4 = 2.7111$$

The use of actual/360 for a money market instrument indicates that the reference period is 360 days. The interest earned during part of a year is calculated by dividing the actual number of elapsed days by 360 and multiplying by the rate. The interest earned in 90 days is therefore exactly one-fourth of the quoted rate. Note that the interest earned in a whole year of 365 days is 365/360 times the quoted rate.

## 5.9 QUOTATIONS

The price quoted for an interest-bearing instrument is often not the same as the cash price you would pay if you purchased it. We illustrate this by considering the way in which prices are quoted for Treasury bonds and Treasury bills in the United States.

### Bonds

Treasury bond prices in the United States are quoted in dollars and thirty-seconds of a dollar. The quoted price is for a bond with a face value of \$100. Thus, a quote of 90-05 indicates that the quoted price for a bond with a face value of \$100,000 is \$90,156.25.

The quoted price is not the same as the cash price that is paid by the purchaser. In general,

$$\text{Cash price} = \text{Quoted price} + \text{Accrued interest since last coupon date}$$

To illustrate this formula, suppose that it is March 5, 2001, and the bond under consideration is an 11% coupon bond maturing on July 10, 2009, with a quoted price of 95-16, or \$95.50. Because coupons are paid semiannually on government bonds, the most recent coupon date is January 10, 2001, and the next coupon date is July 10, 2001. The number of days between January 10, 2001, and March 5, 2001, is 54, whereas the

number of days between January 10, 2001, and July 10, 2001, is 181. On a bond with \$100 face value, the coupon payment is \$5.50 on January 10 and July 10. The accrued interest on March 5, 2001, is the share of the July 10 coupon accruing to the bondholder on March 5, 2001. Because actual/actual in period is used for Treasury bonds, this is

$$\frac{54}{181} \times \$5.5 = \$1.64$$

The cash price per \$100 face value for the July 10, 2001, bond is therefore

$$\$95.5 + \$1.64 = \$97.14$$

Thus, the cash price of a \$100,000 bond is \$97,140.

### Treasury Bills

As already mentioned, the actual/360 day count convention is used for Treasury bills in the United States. Price quotes are for a Treasury bill with a face value of \$100. There is a difference between the cash price and quoted price for a Treasury bill. If  $Y$  is the cash price of a Treasury bill that has a face value of \$100 and  $n$  days to maturity, the quoted price is

$$\frac{360}{n}(100 - Y)$$

This is referred to as the *discount rate*. It is the annualized dollar return provided by the Treasury bill expressed as a percentage of the face value. If for a 91-day Treasury bill the cash price,  $Y$ , were 98, the quoted price would be  $(360/91) \times (100 - 98) = 7.91$ .

The discount rate or quoted price is not the same as the rate of return earned on the Treasury bill. The latter is calculated as the dollar return divided by the cost. In the preceding example, where the cash price is 98, the rate of return would be  $2/98$ , or 2.04% per 91 days. This amounts to

$$\frac{2}{98} \times \frac{365}{91} = 0.08186$$

or 8.186% per annum with compounding every 91 days.

## 5.10 TREASURY BOND FUTURES

Table 5.5 shows interest rate futures quotes as they appeared in the *Wall Street Journal* on March 16, 2001. The most popular long-term interest rate futures contract is the Treasury bond futures contract traded on the Chicago Board of Trade (CBOT). In this contract, any government bond that has more than 15 years to maturity on the first day of the delivery month and that is not callable within 15 years from that day can be delivered. As will be explained later, the CBOT has developed a procedure for adjusting the price received by the party with the short position according to the particular bond delivered.

The Treasury note and five-year Treasury note futures contract in the United States are also actively traded. In the Treasury note futures contract, any government bond (or note) with a maturity between  $6\frac{1}{2}$  and 10 years can be delivered. In the five-year

Table 5.5 Interest rate futures quotes from the Wall Street Journal on March 16, 2001

INTEREST RATE									
<b>Treasury Bonds (CBT)-\$100,000; pts 32nds of 100%</b>									
Mar	106-15	106-26	106-03	106-14	+	3	106-30	89-06	47,482
June	106-06	106-17	106-24	106-04	+	3	106-17	96-21	474,584
Sept	106-15	106-02	105-12	105-22	+	2	106-04	96-22	2,818
Est vol 225,000; vol Wed 328,320; open int 524,994, +6,141.									
<b>Treasury Notes (CBT)-\$100,000; pts 32nds of 100%</b>									
Mar	06-205	107-04	106-18	06-295	+	8.5	107-04	98-04	52,833
June	106-11	106-22	106-03	106-15	+	8.5	106-22	99-11	495,880
Sept	106-07	106-09	106-04	106-04	+	8.5	106-09	103-30	4,939
Est vol 226,000; vol Wed 278,130; open int 553,652, +13,722.									
<b>10 Yr Agency Notes (CBT)-\$100,000; pts 32nds of 100%</b>									
Mar	101-30	102-12	01-295	102-07	+	12.0	102-21	93-25	10,264
June	101-17	102-02	101-14	101-27	+	11.5	02-105	98-22	46,569
Est vol 4,000; vol Wed 9,020; open int 56,833, +573.									
<b>5 Yr Treasury Notes (CBT)-\$100,000; pts 32nds of 100%</b>									
Mar	105-05	05-155	105-01	05-115	+	8.0	05-155	100-11	36,713
June	105-15	05-245	105-09	105-20	+	9.5	05-245	101-04	329,523
Est vol 127,000; vol Wed 117,378; open int 366,236, +5,564.									
<b>2 Yr Treasury Notes (CBT)-\$200,000; pts 32nds of 100%</b>									
Mar	102-24	102-31	102-24	102-29	+	5.7	102-31	11-282	6,647
June	102-28	03-022	02-252	103-04	+	6.2	03-022	101-12	78,842
Est vol 10,100; vol Wed 6,628; open int 95,489, -428.									
<b>30 Day Federal Funds (CBT)-\$5 million; pts of 100%</b>									
Mar	94.740	94.750	94.730	94.745	+	.010	95.810	93.205	21,039
Apr	95.18	95.21	95.16	95.20	+	.04	95.21	93.21	28,536
May	95.30	95.35	95.29	95.35	+	.06	95.35	94.02	6,703
June	95.40	95.51	95.42	95.50	+	.07	95.51	94.38	5,403
July	95.62	95.62	95.55	95.60	+	.08	95.62	95.02	2,045
Est vol 14,000; vol Wed 19,184; open int 63,726, -118.									
<b>Muni Bond Index (CBT)-\$1,000; times Bond Buyer MBI</b>									
Mar	105-15	105-20	105-14	105-17	+	---	105-27	98-03	6,532
June	104-21	104-30	104-16	104-22	+	1	105-00	101-30	11,635
Est vol 850; vol Wed 1,203; open int 18,167, -130.									
Index: Close 104-30; Yield 5.30.									
<b>Treasury Bills (CME)-\$1 mil.; pts of 100%</b>									
	OPEN	HIGH	LOW	SETTLE	CHANGE	YIELD	CHANGE	OPEN	INT.
Mar	95.59	95.64	95.59	95.62	+	.05	4.38	-.05	3,139
June	96.06	96.13	96.06	96.11	+	.09	3.89	-.09	1,491
Est vol 485; vol Wed 780; open int 4,630, -80.									
<b>Libor-1 Mo. (CME)-\$3,000,000; pts of 100%</b>									
Mar	94.95	94.99	94.95	94.99	+	.04	5.01	-.04	29,178
Apr	95.15	95.19	95.13	95.18	+	.05	4.82	-.05	19,819
May	95.32	95.40	95.31	95.38	+	.07	4.62	-.07	9,811
June	95.43	95.54	95.43	95.53	+	.09	4.47	-.09	1,030
July	95.50	95.59	95.49	95.58	+	.08	4.42	-.08	464
Aug	95.40	95.40	95.40	95.61	+	.08	4.39	-.08	204
Sept	95.40	95.40	95.67	95.67	+	.10	4.33	-.10	24
Est vol 5,710; vol Wed 7,033; open int 60,543, -267.									
<b>Eurodollar (CME)-\$1 Million; pts of 100%</b>									
Mar	95.06	95.10	95.05	95.09	+	.02	4.91	-.02	516,751
Apr	95.24	95.29	95.24	95.28	+	.04	4.72	-.04	25,625
May	95.41	95.44	95.39	95.43	+	.06	4.57	-.06	3,558
June	95.43	95.55	95.43	95.53	+	.08	4.47	-.08	659,688
July	95.59	95.59	95.58	95.59	+	.12	4.41	-.12	1,012
Sept	95.50	95.64	95.50	95.62	+	.10	4.38	-.10	705,199
Dec	95.34	95.47	95.32	95.45	+	.10	4.55	-.10	391,832
Mr02	95.26	95.40	95.25	95.37	+	.09	4.63	-.09	375,306
June	95.03	95.17	95.03	95.14	+	.09	4.86	-.09	316,329
Sept	94.84	94.96	94.83	94.93	+	.09	5.07	-.09	252,106
Dec	94.61	94.73	94.60	94.71	+	.09	5.29	-.09	163,620
Mr03	94.57	94.68	94.57	94.66	+	.08	5.34	-.08	125,515
June	94.48	94.59	94.48	94.56	+	.08	5.44	-.08	99,406
Sept	94.40	94.51	94.40	94.48	+	.08	5.52	-.08	95,178
Dec	94.27	94.37	94.26	94.34	+	.08	5.66	-.08	79,656
Mr04	94.27	94.38	94.26	94.34	+	.08	5.66	-.08	76,267
June	94.21	94.30	94.18	94.27	+	.08	5.73	-.08	61,145
Sept	94.12	94.24	94.12	94.20	+	.08	5.80	-.08	60,968
Dec	94.03	94.12	94.00	94.08	+	.08	5.92	-.08	51,646
Mr05	94.01	94.12	94.01	94.09	+	.08	5.91	-.08	36,722
June	93.97	94.06	93.97	94.03	+	.08	5.97	-.08	35,002
Sept	93.91	94.01	93.91	93.97	+	.07	6.03	-.07	33,913
Dec	93.81	93.90	93.81	93.86	+	.07	6.14	-.07	25,415
Mr06	93.82	93.92	93.82	93.88	+	.06	6.12	-.06	19,542
June	93.78	93.87	93.78	93.83	+	.06	6.17	-.06	18,291
Sept	93.75	93.82	93.74	93.79	+	.06	6.21	-.06	17,857
Dec	93.65	93.73	93.64	93.69	+	.05	6.31	-.05	13,136
Mr07	93.64	93.71	93.63	93.72	+	.05	6.28	-.05	11,626
June	93.64	93.71	93.63	93.67	+	.04	6.33	-.04	8,488
Sept	93.64	93.71	93.63	93.67	+	.04	6.37	-.04	7,120
Dec	93.64	93.71	93.63	93.67	+	.04	6.46	-.04	7,780
Mr08	93.55	93.61	93.52	93.57	+	.04	6.43	-.04	5,289
June	93.50	93.57	93.48	93.53	+	.03	6.47	-.03	5,064
Sept	93.51	93.54	93.45	93.49	+	.03	6.51	-.03	3,976

Dec	93.42	93.45	93.36	93.40	+	.03	6.80	-.03	3,626
Mr09	93.42	93.45	93.36	93.40	+	.03	6.80	-.03	3,626
June	93.39	93.42	93.39	93.42	+	.02	6.57	-.02	3,935
Sept	93.35	93.38	93.35	93.38	+	.02	6.61	-.02	3,446
Dec	93.35	93.38	93.35	93.38	+	.02	6.65	-.02	3,417
Mr10	93.29	93.32	93.26	93.31	+	.01	6.74	-.01	2,485
June	93.25	93.28	93.25	93.28	+	.01	6.71	-.01	2,534
Sept	93.22	93.25	93.22	93.25	+	.01	6.75	-.01	2,611
Dec	93.22	93.25	93.22	93.25	+	.01	6.78	-.01	2,694
Est vol 792,400; vol Wed 950,073; open int 4,338,283, +26,797.									
<b>Euro-Yan (CME)-Yan 100,000,000; pts of 100%</b>									
	OPEN	HIGH	LOW	SETTLE	CHANGE	HIGH	LOW	OPEN	INT.
Mar	99.77	99.77	99.77	99.77	---	01	99.79	98.07	30,179
June	99.87	99.88	99.87	99.87	---	01	99.89	98.20	15,401
Sept	99.86	99.87	99.86	99.87	---	01	99.86	98.05	12,051
Dec	99.84	99.85	99.84	99.85	---	01	99.86	97.97	8,025
Mr02	99.84	99.85	99.84	99.85	---	01	99.86	97.97	4,954
June	99.83	99.84	99.83	99.83	---	01	99.85	97.87	1,851
Sept	99.78	99.78	99.77	99.78	---	01	99.79	97.84	1,432
Dec	99.67	99.67	99.67	99.67	---	01	99.68	97.77	169
Est vol 4,309; vol Wed 7,187; open int 74,062, +1,496.									
<b>Short Sterling (LIFFE)-£500,000; pts of 100%</b>									
Mar	94.45	94.51	94.45	94.50	+	.06	95.08	92.55	144,961
June	94.77	94.82	94.77	94.80	+	.04	95.08	92.49	148,885
Sept	94.95	94.98	94.93	94.97	+	.05	95.09	92.41	204,792
Dec	94.89	94.94	94.89	94.93	+	.05	95.07	92.31	86,147
Mr02	94.82	94.86	94.82	94.85	+	.05	95.13	92.34	73,235
June	94.73	94.76	94.72	94.75	+	.04	95.10	92.39	83,128
Sept	94.65	94.69	94.65	94.68	+	.04	95.11	92.38	39,315
Dec	94.60	94.63	94.59	94.62	+	.04	95.11	92.45	24,534
Mr03	94.59	94.61	94.57	94.60	+	.04	94.69	92.49	16,439
June	94.57	94.58	94.55	94.57	+	.03	94.58	92.77	15,276
Sept	94.54	94.56	94.52	94.54	+	.04	94.56	92.90	22,180
Dec	94.50	94.52	94.49	94.51	+	.03	94.52	92.92	5,222
Mr04	94.48	94.48	94.47	94.47	+	.03	94.43	93.01	3,759
June	94.45	94.45	94.45	94.45	+	.03	94.41	93.04	1,872
Sept	94.45	94.45	94.45	94.45	+	.03	94.16	93.35	311
Dec	94.45	94.45	94.45	94.45	+	.03	94.30	93.25	581
Est vol 154,768 vol Wed 213,500 open int 852,507 +9,941.									
<b>Long Gilt (LIFFE) (Decimal)-£50,000; pts of 100%</b>									
Mar	116.60	116.81	116.58	116.89	+	.41	155.74	113.10	6,212
June	117.69	118.17	117.18	118.10	+	.41	118.17	114.90	57,299
Est vol 26,977 vol Wed 32,350 open int 63,511 +3,405.									
<b>3 Month Euribor (LIFFE) Euro 1,000,000; pts of 100%</b>									
Mar	95.26	95.26	95.24	95.25	---	.02	95.96	94.57	279,299
Apr	95.26	95.26	95.26	95.26	---	.02	95.92	95.31	615
June	95.58	95.60	95.56	95.59	+	.02	96.85	94.50	326,977
Sept	95.80	95.83	95.78	95.82	+	.04	96.75	94.44	341,037
Dec	95.74	95.76							

Table 5.5 (continued)

<b>10 Yr. Canadian Govt. Bonds (ME)-C\$100,000</b>									
Mar	....	....	....	105.50	+ 0.10	105.25	101.80	5,505	
June	105.10	105.33	104.91	105.81	+ 0.10	105.40	103.30	61,851	
Est vol 2,838; vol Wed 3,551; open int 67,356, -723.									
<b>10 Yr. Euro Notional Bond (MATIF)-Euros 100,000</b>									
Mar	90.32	90.40	90.13	90.30	+ 0.05	90.55	86.33	175,093	
June	90.42	90.70	90.26	90.45	+ 0.06	90.70	89.19	28,708	
Est vol 221,549; vol Wed 218,960; open int 187,299, +6,100.									
<b>3 Month Euribor (MATIF)-Euros 1,000,000</b>									
Mar	....	....	....	95.24	- 0.02	96.95	94.63	6,053	
June	....	....	....	95.58	+ 0.02	96.85	94.55	3,357	
Sept	....	....	....	95.82	+ 0.03	96.75	94.47	2,021	
Dec	....	....	....	95.75	+ 0.03	96.58	94.37	2,835	
Mr02	....	....	....	95.79	+ 0.03	96.48	94.45	2,617	
June	....	....	....	95.69	+ 0.03	96.29	94.35	308	
Sept	....	....	....	95.60	+ 0.04	96.21	94.27	475	
Dec	....	....	....	95.46	+ 0.03	95.97	94.10	330	
Est vol 0; vol Wed 0; open int 18,059, unch.									
<b>3 Yr. Commonwealth T-Bonds (SFE)-A\$100,000</b>									
Mar	95.44	95.57	95.43	95.54	+ 0.09	95.57	94.25	0	
June	95.50	95.62	95.48	95.55	+ 0.05	95.62	94.94	231,145	
Est vol 399,603; vol Wed 168,037; open int 231,145, -295,318.									
<b>Euro-Yen (SAX)-Yen 100,000,000 pts of 100%</b>									
Mar	99.78	99.79	99.77	99.77	- 0.01	99.79	98.08	98,662	
June	99.88	99.88	99.87	99.87	....	99.88	98.17	111,646	
Sept	99.88	99.88	99.87	99.87	- 0.01	99.88	98.01	78,678	
Dec	99.85	99.86	99.84	99.84	- 0.01	99.86	97.84	51,494	
Mr02	99.86	99.86	99.84	99.84	- 0.01	99.86	98.17	37,541	
June	99.85	99.85	99.83	99.83	- 0.02	99.85	98.11	33,907	
Sept	99.79	99.79	99.77	99.76	- 0.01	99.79	98.35	14,517	
Dec	99.70	99.70	99.68	99.67	- 0.01	99.70	98.28	3,264	
Mr03	....	....	....	99.60	- 0.01	99.54	98.45	2,102	
June	....	....	....	99.51	....	99.54	98.40	2,170	
Sept	....	....	....	99.37	....	99.42	98.34	5,010	
Est vol 25,077; vol Wed 34,493; open int 439,071, +82.									
<b>5 Yr. German Euro-Govt. Bond (EURO-BOBL) (EUREX)-Euro 100,000; pts of 100%</b>									
June	106.60	106.75	106.53	106.74	+ 0.08	106.75	103.50	465,268	
Sept	....	....	....	107.02	+ 0.08	106.65	105.51	9,638	
vol Thu 456,995; open int 479,927, +46,697.									
<b>10 Yr. German Euro-Govt. Bond (EURO-BUND) (EUREX)-Euro 100,000; pts of 100%</b>									
June	109.46	109.67	109.40	109.64	+ 0.08	109.70	106.00	687,796	
Sept	109.64	109.70	109.49	109.70	+ 0.01	109.70	107.96	5,747	
vol Thu 708,063; open int 693,543, +92,384.									
<b>2 Yr. German Euro-Govt. Bond (EURO-SCHATZ) (EUREX)-Euro 100,000; pts of 100%</b>									
June	103.03	103.09	102.97	103.07	+ 0.03	103.22	102.27	507,068	
Sept	....	....	....	102.98	+ 0.03	....	....	4,304	
vol Thu 379,199; open int 511,372, +57,585.									

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Treasury note futures contract, any of the four most recently auctioned Treasury notes can be delivered.

The remaining discussion in this section focuses on Treasury bond futures. Many other bond futures contracts traded in the United States and the rest of the world are designed in a similar way to CBOT Treasury bond futures so that many of the points we will make are applicable to these contracts as well.

## Quotes

Treasury bond futures prices are quoted in the same way as the Treasury bond prices themselves (see Section 5.9). Table 5.5 shows that the settlement price on March 15, 2001, for the June 2001 contract was 106-04, or  $106\frac{4}{32}$ . One contract involves the delivery of \$100,000 face value of the bond. Thus, a \$1 change in the quoted futures price would lead to a \$1,000 change in the value of the futures contract. Delivery can take place at any time during the delivery month.

## Conversion Factors

As mentioned, the Treasury bond futures contract allows the party with the short position to choose to deliver any bond that has a maturity of more than 15 years and that is not callable within 15 years. When a particular bond is delivered, a parameter known as its *conversion factor* defines the price received by the party with the short position. The quoted price applicable to the delivery is the product of the conversion factor and the quoted futures price. Taking accrued interest into account, as described in Section 5.9, the cash received for each \$100 face value of bond delivered is

$$(\text{Quoted futures price} \times \text{Conversion factor}) + \text{Accrued interest}$$

Each contract is for the delivery of \$100,000 face value of bonds. Suppose the quoted futures price is 90-00, the conversion factor for the bond delivered is 1.3800, and the accrued interest on this bond at the time of delivery is \$3 per \$100 face value. The cash received by the party with the short position (and paid by the party with the long

position) is then

$$(1.3800 \times 90.00) + 3.00 = \$127.20$$

per \$100 face value. A party with the short position in one contract would deliver bonds with face value of \$100,000 and receive \$127,200.

The conversion factor for a bond is equal to the value of the bond per dollar of principal on the first day of the delivery month on the assumption that the interest rate for all maturities equals 6% per annum (with semiannual compounding).<sup>5</sup> The bond maturity and the times to the coupon payment dates are rounded down to the nearest three months for the purposes of the calculation. The practice enables the CBOT to produce comprehensive tables. If, after rounding, the bond lasts for an exact number of six months, the first coupon is assumed to be paid in six months. If, after rounding, the bond does not last for an exact number of six months (i.e., there is an extra three months), the first coupon is assumed to be paid after three months and accrued interest is subtracted.

As a first example of these rules, consider a 10% coupon bond with 20 years and two months to maturity. For the purposes of calculating the conversion factor, the bond is assumed to have exactly 20 years to maturity. The first coupon payment is assumed to be made after six months. Coupon payments are then assumed to be made at six-month intervals until the end of the 20 years when the principal payment is made. Assume that the face value is \$100. When the discount rate is 6% per annum with semiannual compounding (or 3% per six months), the value of the bond is

$$\sum_{i=1}^{40} \frac{5}{1.03^i} + \frac{100}{1.03^{40}} = \$146.23$$

Dividing by the face value gives a credit conversion factor of 1.4623.

As a second example of the rules, consider an 8% coupon bond with 18 years and four months to maturity. For the purposes of calculating the conversion factor, the bond is assumed to have exactly 18 years and three months to maturity. Discounting all the payments back to a point in time three months from today at 6% per annum (compounded semiannually) gives a value of

$$\sum_{i=0}^{36} \frac{4}{1.03^i} + \frac{100}{1.03^{36}} = \$125.83$$

The interest rate for a three-month period is  $\sqrt{1.03} - 1$ , or 1.4889%. Hence, discounting back to the present gives the bond's value as  $125.83/1.014889 = \$123.99$ . Subtracting the accrued interest of 2.0, this becomes \$121.99. The conversion factor is therefore 1.2199.

### Cheapest-to-Deliver Bond

At any given time during the delivery month, there are many bonds that can be delivered in the CBOT Treasury bond futures contract. These vary widely as far as coupon and maturity are concerned. The party with the short position can choose which of the available bonds is "cheapest" to deliver. Because the party with the short

<sup>5</sup> For contracts with maturities prior to March 2000, this interest rate was 8% in the CBOT conversion factor calculation.

**Table 5.6** Deliverable bonds in Example

Bond	Quoted price (\$)	Conversion factor
1	99.50	1.0382
2	143.50	1.5188
3	119.75	1.2615

position receives

$$(\text{Quoted futures price} \times \text{Conversion factor}) + \text{Accrued interest}$$

and the cost of purchasing a bond is

$$\text{Quoted price} + \text{Accrued interest}$$

the cheapest-to-deliver bond is the one for which

$$\text{Quoted price} - (\text{Quoted futures price} \times \text{Conversion factor})$$

is least. Once the party with the short position has decided to deliver, it can determine the cheapest-to-deliver bond by examining each of the bonds in turn.

### Example

The party with the short position has decided to deliver and is trying to choose among the three bonds in Table 5.6. Assume the current quoted futures price is 93-08, or 93.25. The cost of delivering each of the bonds is as follows:

$$\text{Bond 1: } 99.50 - (93.25 \times 1.0382) = \$2.69$$

$$\text{Bond 2: } 143.50 - (93.25 \times 1.5188) = \$1.87$$

$$\text{Bond 3: } 119.75 - (93.25 \times 1.2615) = \$2.12$$

The cheapest-to-deliver bond is bond 2.

A number of factors determine the cheapest-to-deliver bond. When bond yields are in excess of 6%, the conversion factor system tends to favor the delivery of low-coupon, long-maturity bonds. When yields are less than 6%, the system tends to favor the delivery of high-coupon, short-maturity bonds. Also, when the yield curve is upward sloping, there is a tendency for bonds with a long time to maturity to be favored, whereas when it is downward sloping, there is a tendency for bonds with a short time to maturity to be delivered.

### The Wild Card Play

Trading in the CBOT Treasury bond futures contract ceases at 2:00 p.m. Chicago time. However, Treasury bonds themselves continue trading in the spot market until 4:00 p.m. Furthermore, a trader with a short futures position has until 8:00 p.m. to issue to the clearinghouse a notice of intention to deliver. If the notice is issued, the invoice price is calculated on the basis of the settlement price that day. This is the price at which trading was conducted just before the closing bell at 2:00 p.m.

This practice gives rise to an option known as the *wild card play*. If bond prices decline after 2:00 p.m., the party with the short position can issue a notice of intention to deliver

and proceed to buy cheapest-to-deliver bonds in preparation for delivery at the 2:00 p.m. futures price. If the bond price does not decline, the party with the short position keeps the position open and waits until the next day when the same strategy can be used.

As with the other options open to the party with the short position, the wild card play is not free. Its value is reflected in the futures price, which is lower than it would be without the option.

### Determining the Quoted Futures Price

An exact theoretical futures price for the Treasury bond contract is difficult to determine because the short party's options concerned with the timing of delivery and choice of the bond that is delivered cannot easily be valued. However, if we assume that both the cheapest-to-deliver bond and the delivery date are known, the Treasury bond futures contract is a futures contract on a security providing the holder with known income.<sup>6</sup> Equation (3.6) then shows that the futures price,  $F_0$ , is related to the spot price,  $S_0$ , by

$$F_0 = (S_0 - I)e^{rT} \quad (5.5)$$

where  $I$  is the present value of the coupons during the life of the futures contract,  $T$  is the time until the futures contract matures, and  $r$  is the risk-free interest rate applicable to a time period of length  $T$ .

### Example

Suppose that, in a Treasury bond futures contract, it is known that the cheapest-to-deliver bond will be a 12% coupon bond with a conversion factor of 1.4000. Suppose also that it is known that delivery will take place in 270 days. Coupons are payable semiannually on the bond. As illustrated in Figure 5.2, the last coupon date was 60 days ago, the next coupon date is in 122 days, and the coupon date thereafter is in 305 days. The term structure is flat, and the rate of interest (with continuous compounding) is 10% per annum. Assume that the current quoted bond price is \$120. The cash price of the bond is obtained by adding to this quoted price the proportion of the next coupon payment that accrues to the holder. The cash price is therefore

$$120 + \frac{60}{60 + 122} \times 6 = 121.978$$

A coupon of \$6 will be received after 122 days (= 0.3342 year). The present value of this is

$$6e^{-0.1 \times 0.3342} = 5.803$$

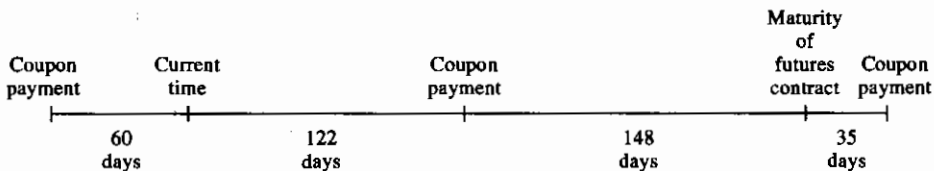


Figure 5.2 Time chart for Example

<sup>6</sup> In practice, for the purposes of determining the cheapest-to-deliver in this calculation, analysts usually assume that zero rates at the maturity of the futures contract will equal today's forward rates.

The futures contract lasts for 270 days (0.7397 year). The cash futures price if the contract were written on the 12% bond would therefore be

$$(121.978 - 5.803)e^{0.1 \times 0.7397} = 125.094$$

At delivery there are 148 days of accrued interest. The quoted futures price if the contract were written on the 12% bond is calculated by subtracting accrued interest:

$$125.094 - 6 \times \frac{148}{305 - 122} = 120.242$$

The conversion factor means that 1.4000 standard bonds are considered equivalent to each 12% bond. The quoted futures price should therefore be

$$\frac{120.242}{1.4000} = 85.887$$

## 5.11 EURODOLLAR FUTURES

Another very popular United States interest rate futures contract is the three-month Eurodollar futures contract traded on the Chicago Mercantile Exchange (CME). A Eurodollar is a dollar deposited in a U.S. or foreign bank outside the United States. The Eurodollar interest rate is the rate of interest earned on Eurodollars deposited by one bank with another bank. It is essentially the same as the London Interbank Offer Rate (LIBOR) mentioned in earlier chapters.

Three-month Eurodollar futures contracts are futures contracts on the three-month Eurodollar interest rate. The contracts have maturities in March, June, September, and December for up to 10 years into the future. In addition, as can be seen from Table 5.5, the CME trades short-maturity contracts with maturities in other months.

If  $Q$  is the quoted price for a Eurodollar futures contract, the exchange defines

$$10,000[100 - 0.25(100 - Q)] \quad (5.6)$$

as the value of one contract. Thus, the settlement price of 95.53 for the June 2001 contract in Table 5.5 corresponds to a contract price of

$$10,000[100 - 0.25(100 - 95.53)] = \$988,825$$

It can be seen from equation (5.6) that a change of one basis point or 0.01 in a Eurodollar futures quote corresponds to a contract price change of \$25.

When the third Wednesday of the delivery month is reached the contract is settled in cash. The final marking to market sets  $Q$  equal to  $100 - R$ , where  $R$  is the actual three-month Eurodollar interest rate on that day, expressed with quarterly compounding and an actual/360 day count convention. Thus, if the three-month Eurodollar interest rate on the third Wednesday of the delivery month is 8%, the final marking to market is 92, and the final contract price is from equation (5.6)

$$10,000[100 - 0.25(100 - 92)] = \$980,000$$

If  $Q$  is a Eurodollar futures quote,  $(100 - Q)\%$  is the Eurodollar futures interest rate for a three-month period beginning on the third Wednesday of the delivery month. Thus Table 5.5 indicates that on March 15, 2001, the futures interest rate for the three-

month period beginning Wednesday June 20, 2001, was  $100 - 95.53 = 4.47\%$ . This is expressed with quarterly compounding and an actual/360 day count convention.

Other contracts similar to the CME Eurodollar futures contracts trade on interest rates in other countries. As shown in Table 5.5, the CME and SGX trade Euroyen contracts, LIFFE and MATIF trade Euribor contracts (i.e., contracts on the three-month LIBOR rate for the euro), and LIFFE trades three-month Euroswiss futures. (See Table 2.2 for the meaning of these exchange abbreviations.)

### Forward vs. Futures Interest Rates

For short maturities (up to one year) the Eurodollar futures interest rate can be assumed to be the same as the corresponding forward interest rate. For longer-dated contracts the differences between futures and forward interest rates mentioned in Section 3.9 become important.

Analysts make what is known as a *convexity adjustment* to convert Eurodollar futures rates to forward interest rates. One way of doing this is by using the formula

$$\text{Forward rate} = \text{Futures rate} - \frac{1}{2}\sigma^2 t_1 t_2$$

where  $t_1$  is the time to maturity of the futures contract,  $t_2$  is the time to the maturity of the rate underlying the futures contract, and  $\sigma$  is the standard deviation of the change in the short-term interest rate in one year. Both rates are expressed with continuous compounding.<sup>7</sup> A typical value for  $\sigma$  is 1.2% or 0.012.

### Example

Consider the situation where  $\sigma = 0.012$ , and we wish to calculate the forward rate when the eight-year Eurodollar futures price quote is 94. In this case  $t_1 = 8$ ,  $t_2 = 8.25$ , and the convexity adjustment is

$$\frac{1}{2} \times 0.012^2 \times 8 \times 8.25 = 0.00475$$

or 0.475%. The futures rate is 6% per annum on an actual/360 basis with quarterly compounding. This is  $6 \times 365/360 = 6.083\%$  per annum on an actual/365 basis with quarterly compounding or 6.038% with continuous compounding. The forward rate is, therefore,  $6.038 - 0.475 = 5.563\%$  per annum with continuous compounding.

The forward rate is less than the futures rate.<sup>8</sup> The size of the adjustment is roughly proportional to the square of the time to maturity of the futures contract. Thus the convexity adjustment for the eight-year contract is approximately 64 times that for a one-year contract.

### Using Eurodollar Futures to Determine LIBOR Zero Rates

Spot LIBOR rates are usually used to determine the LIBOR zero curve out to one year. Beyond one year Eurodollar futures contract are frequently used. Once a convexity adjustment such as that just described is made, the Eurodollar futures contracts define

<sup>7</sup> This formula is based on an interest rate model known as the Ho-Lee model.

<sup>8</sup> The reason for this can be seen from the arguments in Section 3.9. The variable underlying the Eurodollar futures contracts is an interest rate and therefore is highly positively correlated to other interest rates.

forward rates for future three-month time periods. Suppose that the  $i$ th Eurodollar futures contract matures at time  $T_i$  ( $i = 1, 2, \dots$ ). It is usually assumed that the forward interest rate calculated from this futures contract applies to the period  $T_i$  to  $T_{i+1}$ . This enables a bootstrap procedure to be used to determine zero rates. Suppose that  $F_i$  is the forward rate calculated from the  $i$ th Eurodollar futures contract and  $R_i$  is the zero rate for a maturity  $T_i$ . From equation (5.1),

$$F_i = \frac{R_{i+1}T_{i+1} - R_iT_i}{T_{i+1} - T_i}$$

so that

$$R_{i+1} = \frac{F_i(T_{i+1} - T_i) + R_iT_i}{T_{i+1}} \quad (5.7)$$

### Example

The 400-day LIBOR zero rate has been calculated as 4.80% with continuous compounding and, from a Eurodollar futures quote, it has been calculated that the forward rate for a 91-day period beginning in 400 days is 5.30% with continuous compounding. We can use equation (5.7) to obtain the 491-day rate as

$$\frac{0.053 \times 91 + 0.048 \times 400}{491} = 0.04893$$

or 4.893%.

## 5.12 DURATION

The *duration* of a bond, as its name implies, is a measure of how long on average the holder of the bond has to wait before receiving cash payments. A zero-coupon bond that matures in  $n$  years has a duration of  $n$  years. However, a coupon-bearing bond maturing in  $n$  years has a duration of less than  $n$  years, because the holder receives some of the cash payments prior to year  $n$ .

Suppose that a bond provides the holder with cash flows  $c_i$  at time  $t_i$  ( $1 \leq i \leq n$ ). The price,  $B$ , and yield,  $y$  (continuously compounded), are related by

$$B = \sum_{i=1}^n c_i e^{-yt_i} \quad (5.8)$$

The duration,  $D$ , of the bond is defined as

$$D = \frac{\sum_{i=1}^n t_i c_i e^{-yt_i}}{B} \quad (5.9)$$

This can be written

$$D = \sum_{i=1}^n t_i \left[ \frac{c_i e^{-yt_i}}{B} \right]$$

The term in square brackets is the ratio of the present value of the cash flow at time  $t_i$  to the bond price. The bond price is the present value of all payments. The duration is therefore a weighted average of the times when payments are made, with the weight applied to time  $t_i$  being equal to the proportion of the bond's total present value provided by the cash flow at time  $t_i$ . The sum of the weights is 1.0.

**Table 5.7** Calculation of duration

Time (years)	Cash flow (\$)	Present value	Weight	Time × Weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total	130	94.213	1.000	2.653

From equation (5.8), it is approximately true that

$$\delta B = -\delta y \sum_{i=1}^n c_i t_i e^{-y t_i} \quad (5.10)$$

where  $\delta y$  is a small change in  $y$  and  $\delta B$  is the corresponding small change in  $B$ . (Note that there is a negative relationship between  $B$  and  $y$ . When bond yields increase, bond prices decrease. When bond yields decrease, bond prices increase.) From equations (5.9) and (5.10),

$$\delta B = -BD \delta y \quad (5.11)$$

or

$$\frac{\delta B}{B} = -D \delta y \quad (5.12)$$

Equation (5.12) is an approximate relationship between proportional changes in a bond price and changes in its yield. It is the key duration relationship. It explains why duration is such a popular and widely used measure.

Consider a three-year 10% coupon bond with a face value of \$100. Suppose that the yield on the bond is 12% per annum with continuous compounding. This means that  $y = 0.12$ . Coupon payments of \$5 are made every six months. Table 5.7 shows the calculations necessary to determine the bond's duration. The present values of the bond's cash flows, using the yield as the discount rate, are shown in column 3. (For example, the present value of the first cash flow is  $5e^{-0.12 \times 0.5} = 4.709$ .) The sum of the numbers in column 3 gives the bond's price as 94.213. The weights are calculated by dividing the numbers in column 3 by 94.213. The sum of the numbers in column 5 gives the duration as 2.653 years.

Small changes in interest rates are often measured in *basis points*. A basis point is 0.01% per annum. The following example investigates the accuracy of the duration relationship in equation (5.11).

### Example

For the bond in Table 5.7, the bond price,  $B$ , is 94.213 and the duration,  $D$ , is 2.653 so that equation (5.11) gives

$$\delta B = -94.213 \times 2.653 \delta y$$

or

$$\delta B = -249.95 \delta y$$

When the yield on the bond increases by 10 basis points ( $= 0.1\%$ ),  $\delta y = +0.001$ . The duration relationship predicts that  $\delta B = -249.95 \times 0.001 = -0.250$  so that the bond price goes down to  $94.213 - 0.250 = 93.963$ . How accurate is this? When the bond yield increases by 10 basis points to 12.1%, the bond price is

$$5e^{-0.121 \times 0.5} + 5e^{-0.121} + 5e^{-0.121 \times 1.5} \\ + 5e^{-0.121 \times 2.0} + 5e^{-0.121 \times 2.5} + 105e^{-0.121 \times 1.5} = 93.963$$

which is (to three decimal places) the same as the bond price predicted by the duration relationship.

### Modified Duration

The preceding analysis is based on the assumption that  $y$  is expressed with continuous compounding. If  $y$  is expressed with annual compounding, it can be shown that the approximate relationship in equation (5.11) becomes

$$\delta B = -\frac{BD \delta y}{1 + y}$$

More generally, if  $y$  is expressed with a compounding frequency of  $m$  times per year,

$$\delta B = -\frac{BD \delta y}{1 + y/m}$$

A variable  $D^*$  defined by

$$D^* = \frac{D}{1 + y/m}$$

is sometimes referred to as the bond's *modified duration*. It allows the duration relationship to be simplified to

$$\delta B = -BD^* \delta y \quad (5.13)$$

when  $y$  is expressed with a compounding frequency of  $m$  times per year. The following example investigates the accuracy of the modified duration relationship.

### Example

The bond in Table 5.7 has a price of 94.213 and a duration of 2.653. The yield, expressed with semiannual compounding is 12.3673%. The modified duration,  $D^*$  is

$$D^* = \frac{2.653}{1 + 0.123673/2} = 2.499$$

From equation (5.13),

$$\delta B = -94.213 \times 2.4985 \delta y$$

or

$$\delta B = -235.39 \delta y$$

When the yield (semiannually compounded) increases by 10 basis points ( $= 0.1\%$ ),  $\delta y = +0.001$ . The duration relationship predicts that we expect  $\delta B$  to be  $-235.39 \times 0.001 = -0.235$  so that the bond price goes down to

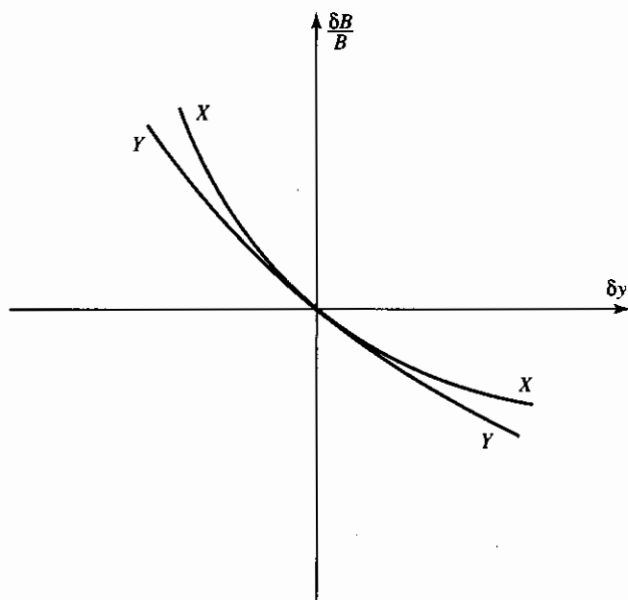
$94.213 - 0.235 = 93.978$ . How accurate is this? When the bond yield (semiannually compounded) increases by 10 basis points to 12.4673% (or to 12.0941% with continuous compounding), an exact calculation similar to that in the previous example shows that the bond price becomes 93.978. This shows that the modified duration calculation gives good accuracy.

### Bond Portfolios

The duration,  $D$ , of a bond portfolio can be defined as a weighted average of the durations of the individual bonds in the portfolio, with the weights being proportional to the bond prices. Equations (5.11) to (5.13) then apply with  $B$  being defined as the value of the bond portfolio. They estimate the change in the value of the bond portfolio for a particular change  $\delta y$  in the yields of all the bonds.

It is important to realize that, when duration is used for bond portfolios, there is an implicit assumption that the yields of all bonds will change by the same amount. When the bonds have widely differing maturities, this happens only when there is a parallel shift in the zero-coupon yield curve. We should therefore interpret equations (5.11) to (5.13) as providing estimates of the impact on the price of a bond portfolio of a parallel shift,  $\delta y$ , in the zero curve.

The duration relationship applies only to small changes in yields. This is illustrated in Figure 5.3, which shows the relationship between the percentage change in value and change in yield for two bond portfolios having the same duration. The gradients of the two curves are the same at the origin. This means that both bond portfolios change in value by the same percentage for small yield changes and is consistent with equation (5.12). For large yield changes, the portfolios behave differently. Portfolio  $X$  has more curvature in its relationship with yields than portfolio  $Y$ . A factor known as *convexity* measures this curvature and can be used to improve the relationship in equation (5.12).



**Figure 5.3** Two bond portfolios with the same duration

### Hedging Portfolios of Assets and Liabilities

Financial institutions frequently attempt to hedge themselves against interest-rate risk by ensuring that the average duration of their assets equals the average duration of their liabilities. (The liabilities can be regarded as short positions in bonds.) This strategy is known as *duration matching* or *portfolio immunization*. When implemented it ensures that a small parallel shift in interest rates will have little effect on the value of the portfolio of assets and liabilities. The gain (loss) on the assets should offset the loss (gain) on the liabilities.

Duration matching does not immunize a portfolio against nonparallel shifts in the zero curve. This is a weakness of the approach. In practice, short-term rates are usually more volatile than and are not perfectly correlated with long-term rates. Sometimes it even happens that short- and long-term rates move in opposite directions to each other. Financial institutions often try to allow for nonparallel shifts by dividing the zero-coupon yield curve up into segments and ensuring that they are hedged against a movement in each segment. Suppose that the  $i$ th segment is the part of the zero-coupon yield curve between maturities  $t_i$  and  $t_{i+1}$ . A financial institution would examine the effect of a small increase  $\delta y$  in all the zero rates with maturities between  $t_i$  and  $t_{i+1}$  while keeping the rest of the zero-coupon yield curve unchanged. If the exposure were unacceptable, further trades would be undertaken in carefully selected instruments to reduce it. In the context of a bank managing a portfolio of assets and liabilities, this approach is sometimes referred to as *GAP management*.

## 5.13 DURATION-BASED HEDGING STRATEGIES

Consider the situation where a position in an interest-rate-dependent asset such as a bond portfolio or a money market security is being hedged using an interest rate futures contract. Define:

$F_C$ : Contract price for the interest rate futures contract

$D_F$ : Duration of the asset underlying the futures contract at the maturity of the futures contract

$P$ : Forward value of the portfolio being hedged at the maturity of the hedge. In practice this is usually assumed to be the same as the value of the portfolio today

$D_P$ : Duration of the portfolio at the maturity of the hedge

If we assume that the change in the yield,  $\delta y$ , is the same for all maturities, which means that only parallel shifts in the yield curve can occur, it is approximately true that

$$\delta P = -PD_P \delta y$$

To a reasonable approximation, it is also true that

$$\delta F_C = -F_C D_F \delta y$$

The number of contracts required to hedge against an uncertain  $\delta y$  is, therefore,

$$N^* = \frac{PD_P}{F_C D_F} \quad (5.14)$$

This is the *duration-based hedge ratio*. It is sometimes also called the *price sensitivity hedge ratio*.<sup>9</sup> Using it has the effect of making the duration of the entire position zero.

When the hedging instrument is a Treasury bond futures contract, the hedger must base  $D_F$  on an assumption that one particular bond will be delivered. This means that the hedger must estimate which of the available bonds is likely to be cheapest to deliver at the time the hedge is put in place. If, subsequently, the interest rate environment changes so that it looks as though a different bond will be cheapest to deliver, the hedge has to be adjusted, and its performance may be worse than anticipated.

When hedges are constructed using interest rate futures, it is important to bear in mind that interest rates and futures prices move in opposite directions. When interest rates go up, an interest rate futures price goes down. When interest rates go down, the reverse happens, and the interest rate futures price goes up. Thus, a company in a position to lose money if interest rates drop should hedge by taking a long futures position. Similarly, a company in a position to lose money if interest rates rise should hedge by taking a short futures position.

The hedger tries to choose the futures contract so that the duration of the underlying asset is as close as possible to the duration of the asset being hedged. Eurodollar futures tend to be used for exposures to short-term interest rates, whereas Treasury bond and Treasury note futures contracts are used for exposures to longer-term rates.

### Hedging a Bond Portfolio

Suppose it is August 2. A fund manager has \$10 million invested in government bonds and is concerned that interest rates are expected to be highly volatile over the next three months. The fund manager decides to use the December Treasury bond futures contract to hedge the value of the portfolio. The current futures price is 93-02, or 93.0625. Because each contract is for the delivery of \$100,000 face value of bonds, the futures contract price is \$93,062.50.

The duration of the bond portfolio in three months is 6.8 years. The cheapest-to-deliver bond in the Treasury bond contract is expected to be a 20-year, 12% per annum coupon bond. The yield on this bond is currently 8.8% per annum, and the duration will be 9.2 years at maturity of the futures contract. This example is summarized in Table 5.8.

The fund manager requires a short position in Treasury bond futures to hedge the bond portfolio. If interest rates go up, a gain will be made on the short futures position, and a loss will be made on the bond portfolio. If interest rates decrease, a loss will be made on the short position, but there will be a gain on the bond portfolio. The number of bond futures contracts that should be shorted can be calculated from equation (5.14) as

$$\frac{10,000,000}{93,062.50} \times \frac{6.80}{9.20} = 79.42$$

Rounding to the nearest whole number, the portfolio manager should short 79 contracts.

Suppose that during the period from August 2 to November 2, interest rates decline rapidly and the value of the bond portfolio increases from \$10 million to \$10,450,000.

<sup>9</sup> For a more detailed discussion of equation (5.14), see R. Rendleman, "Duration-Based Hedging with Treasury Bond Futures," *Journal of Fixed Income* 9(1) (June 1999): 84-91.

**Table 5.8** Hedging a bond portfolio*From the Trader's Desk—August 2*

A fund manager responsible for a \$10 million bond portfolio is concerned that interest rates are expected to be highly volatile over the next three months. The fund manager decides to use Treasury bond futures to hedge the value of the bond portfolio. The quoted price for the December Treasury bond futures contract is 93-02. This means that the contract price is \$93,062.50.

*The Strategy*

1. Short 79 December Treasury bond futures contracts on August 2.
2. Close out the position on November 2.

*The Result*

During the period August 2 to November 2, interest rates declined rapidly. The value of the bond portfolio increased from \$10 million to \$10,450,000.

On November 2, the Treasury bond futures price was 98-16. This corresponds to a contract price of \$98,500.00. A loss of  $79 \times (\$98,500.00 - \$93,062.50) = \$429,562.50$  was therefore made on the Treasury bond futures contracts.

Overall, the value of the portfolio manager's position changed by only  $\$450,000.00 - \$429,562.50 = \$20,437.50$ .

Suppose further that on November 2, the Treasury bond futures price is 98-16. This corresponds to a contract price of \$98,500. The total loss on the Treasury bond futures contracts is

$$79 \times (\$98,500.00 - \$93,062.50) = \$429,562.50$$

The net change in the value of the portfolio manager's position is therefore only

$$\$450,000.00 - \$429,562.50 = \$20,437.50$$

Because the fund incurs a loss on the futures position, the manager may regret implementing the hedge. On average, we can expect half of our hedges to lead to these sorts of regrets. Unfortunately, we do not know in advance which half of the hedges it will be!

### Hedging a Floating-Rate Loan

Interest rate futures can be used to hedge the rate of interest paid by a borrower on a floating-rate loan. Eurodollar futures are ideal for this because the Eurodollar interest rate is closely related to the rate of interest at which corporations borrow. We will consider the use of Eurodollar futures to hedge a three-month loan in which the interest rate is reset every month. This will produce a simple example. The same principles can be used for loans that last far longer than three months.

We suppose that it is April 29, and a company has just borrowed \$15 million for three months. The interest rate for each of the three one-month periods will be the one-month LIBOR rate plus 1%. At the time the loan is negotiated, the one-month LIBOR rate is 8% per annum so that the company must pay 9% per annum for the first month. Because the one-month LIBOR rate is quoted with monthly compounding, the interest

**Table 5.9** Hedging a floating-rate loan*From the Trader's Desk—April 29*

A company has just borrowed \$15 million for three months at an interest rate equal to one-month LIBOR plus 1% and would like to hedge its risk. Quotes:

1. The one-month LIBOR rate is 8%.
2. The June Eurodollar futures price is 91.88.
3. The September Eurodollar futures price is 91.44.

*The Strategy*

1. Short five June contracts and five September contracts.
2. Close out the June contracts on May 29.
3. Close out the September contracts on June 29.

*The Result*

On May 29 the one-month LIBOR rate was 8.8%, and the June futures price was 91.12. The company gained  $5 \times (\$979,700 - \$977,800) = \$9,500$  on the five June contracts. This provided compensation for the \$10,000 extra interest payment necessary in the second month because of the increase in LIBOR from 8% to 8.8%.

On June 29 the one-month LIBOR rate was 9.4% and the September futures price was 90.16. The company gained \$16,000 on the five September contracts. This provided compensation for extra interest costs of \$17,500.

for the first month is 0.75% of \$15 million, or \$112,500. This is known for certain at the time the loan is negotiated and does not have to be hedged.

The interest paid at the end of the second month is determined by the one-month LIBOR rate at the beginning of the second month. It can be hedged by taking a position in the June Eurodollar futures contract. Suppose that the quoted price for this contract is 91.88. From Section 5.11, the contract price is

$$10,000[100 - 0.25(100 - 91.88)] = \$979,700$$

The company will lose money if interest rates rise and gain if interest rates fall. It therefore requires a short position in the futures contracts. The duration of the asset underlying the futures contract at maturity of the contract is three months, or 0.25 years. The duration of the liability being hedged is one month, or 0.08333 years. From equation (5.14), the number of contracts that should be used to hedge the interest payment in the second month is

$$\frac{0.08333}{0.25} \times \frac{15,000,000}{979,700} = 5.10$$

Rounding to the nearest whole number, five contracts are required.

For the third month, the September Eurodollar futures contract can be used. Suppose the quoted price for this contract is 91.44, which corresponds to a futures price of \$978,600. The number of futures contracts that should be shorted can be calculated as before:

$$\frac{0.08333}{0.25} \times \frac{15,000,000}{978,600} = 5.11$$

Again, we find that, to the nearest whole number, five contracts are required. Thus, five

of the June contracts should be shorted to hedge the LIBOR rate applicable to the second month, and five of the September contracts should be shorted to hedge the LIBOR rate applicable to the third month. The June contracts are closed out on May 29, and the September contracts are closed out on June 29.

Suppose that on May 29 the one-month LIBOR rate is 8.8% and the June futures price is 91.12. The latter corresponds to a contract price of \$977,800, so that the company makes a profit of

$$5 \times (\$979,700 - \$977,800) = \$9,500$$

on the June contracts. This provided compensation for the extra \$10,000 interest (one-twelfth of 0.8% of \$15 million) that had to be paid at the end of the second month as a result of the LIBOR increase from 8% to 8.8%.

Suppose further that on June 29 the one-month LIBOR rate is 9.4% and the September futures price is 90.16. A similar calculation to that just given shows that the company gains \$16,000 on the short futures position, but incurs extra interest costs of \$17,500 as a result of the increase in one-month LIBOR from 8% per annum to 9.4% per annum. This example is summarized in Table 5.9.

## 5.14 SUMMARY

Many different interest rates are quoted in financial markets and calculated by analysts. The  $n$ -year zero rate or  $n$ -year spot rate is the rate applicable to an investment lasting for  $n$  years when all of the return is realized at the end. Forward rates are rates applicable to futures periods of time implied by today's zero rates. The par yield on a bond of a certain maturity is the coupon rate that causes the bond to sell for its par value.

Treasury rates should be distinguished from LIBOR rates. Treasury rates are the rates at which the government of the country borrows and are often referred to as risk-free rates. LIBOR rates are the rates at which one large international bank is prepared to lend to another large international bank.

Two very popular interest rate contracts are the Treasury bond and Eurodollar futures contracts that trade in the United States. In the Treasury bond futures contracts, the party with the short position has a number of interesting delivery options:

1. Delivery can be made on any day during the delivery month.
2. There are a number of alternative bonds that can be delivered.
3. On any day during the delivery month, the notice of intention to deliver at the 2:00 p.m. settlement price can be made any time up to 8:00 p.m.

These options all tend to reduce the futures price.

The Eurodollar futures contract is a contract on the three-month rate starting on the third Wednesday of the delivery month. Eurodollar futures are frequently used to estimate LIBOR forward rates for the purpose of constructing a LIBOR zero curve. When long-dated contracts are used in this way, it is important to make what is termed a convexity adjustment to allow for the marking to market in the futures contract.

The concept of duration is important in hedging interest rate risk. Duration measures how long on average an investor has to wait before receiving payments. It is a weighted

average of the times until payments are received, with the weight for a particular payment time being proportional to the present value of the payment.

A key result underlying the duration-based hedging scheme described in this chapter is

$$\delta B = -BD \delta y$$

where  $B$  is a bond price,  $D$  is its duration,  $\delta y$  is a small change in its yield (continuously compounded), and  $\delta B$  is the resultant small change in  $B$ . The equation enables a hedger to assess the sensitivity of a bond price to small changes in its yield. It also enables the hedger to assess the sensitivity of an interest rate futures price to small changes in the yield of the underlying bond. If the hedger is prepared to assume that  $\delta y$  is the same for all bonds, the result enables the hedger to calculate the number of futures contracts necessary to protect a bond or bond portfolio against small changes in interest rates.

The key assumption underlying the duration-based hedging scheme is that all interest rates change by the same amount. This means that only parallel shifts in the term structure are allowed for. In practice, short-term interest rates are generally more volatile than are long-term interest rates, and hedge performance is liable to be poor if the duration of the bond underlying the futures contract differs markedly from the duration of the asset being hedged.

## Suggestions for Further Reading

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- Gay, G. D., R. W. Kolb, and R. Chiang. "Interest Rate Hedging: An Empirical Test of Alternative Strategies." *Journal of Financial Research* 6 (Fall 1983): 187-97.
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- Resnick, B. G. "The Relationship between Futures Prices for U.S. Treasury Bonds." *Review of Research in Futures Markets* 3 (1984): 88-104.
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- Veit, W. T., and W. W. Reiff. "Commercial Banks and Interest Rate Futures: A Hedging Survey." *Journal of Futures Markets* 3 (1983): 283-93.

### Quiz (Answers at End of Book)

- 5.1. It is January 9, 2001. The price of a Treasury bond with a 12% coupon that matures on October 12, 2009, is quoted as 102-07. What is the cash price?
- 5.2. Suppose that zero interest rates with continuous compounding are as follows:

Maturity (years)	Rate (% per annum)
1	8.0
2	7.5
3	7.2
4	7.0
5	6.9

Calculate forward interest rates for the second, third, fourth, and fifth years.

- 5.3. The term structure is upward sloping. Put the following in order of magnitude:
- The five-year zero rate
  - The yield on a five-year coupon-bearing bond
  - The forward rate corresponding to the period between 5 and 5.25 years in the future
- What is the answer to this question when the term structure is downward sloping?
- 5.4. The six-month and one-year zero rates are both 10% per annum. For a bond that lasts 18 months and pays a coupon of 8% per annum (with a coupon payment having just been made), the yield is 10.4% per annum. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.
- 5.5. The price of a 90-day Treasury bill is quoted as 10.00. What continuously compounded return (on an actual/365 basis) does an investor earn on the Treasury bill for the 90-day period?
- 5.6. What assumptions does a duration-based hedging scheme make about the way in which the term structure moves?
- 5.7. It is January 30. You are managing a bond portfolio worth \$6 million. The duration of the portfolio in six months will be 8.2 years. The September Treasury bond futures price is currently 108-15, and the cheapest-to-deliver bond will have a duration of 7.6 years in September. How should you hedge against changes in interest rates over the next six months?

### Questions and Problems (Answers in Solutions Manual)

- 5.8. Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are 4%, 4.2%, 4.4%, 4.6%, and 4.8% per annum with continuous compounding respectively. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months pays a coupon of 4% per annum semiannually.
- 5.9. A three-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond yield?
- 5.10. Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5%, and 7% respectively. What is the two-year par yield?
- 5.11. Explain carefully why liquidity preference theory is consistent with the observation that the term structure tends to be upward sloping more often than it is downward sloping.

- 5.12. Suppose that zero interest rates with continuous compounding are as follows:

Maturity (years)	Rate (% per annum)
1	12.0
2	13.0
3	13.7
4	14.2
5	14.5

Calculate forward interest rates for the second, third, fourth, and fifth years.

- 5.13. Suppose that zero interest rates with continuous compounding are as follows:

Maturity (months)	Rate (% per annum)
3	8.0
6	8.2
9	8.4
12	8.5
15	8.6
18	8.7

Calculate forward interest rates for the second, third, fourth, fifth, and sixth quarters.

- 5.14. The cash prices of six-month and one-year Treasury bills are 94.0 and 89.0. A 1.5-year bond that will pay coupons of \$4 every six months currently sells for \$94.84. A two-year bond that will pay coupons of \$5 every six months currently sells for \$97.12. Calculate the six-month, one-year, 1.5-year, and two-year zero rates.
- 5.15. A 10-year, 8% coupon bond currently sells for \$90. A 10-year, 4% coupon bond currently sells for \$80. What is the 10-year zero rate? (Hint: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.)
- 5.16. It is May 5, 2003. The quoted price of a government bond with a 12% coupon that matures on July 27, 2011, is 110-17. What is the cash price?
- 5.17. Suppose that the Treasury bond futures price is 101-12. Which of the following four bonds is cheapest to deliver?

Bond	Price	Conversion factor
1	125-05	1.2131
2	142-15	1.3792
3	115-31	1.1149
4	144-02	1.4026

- 5.18. It is July 30, 2001. The cheapest-to-deliver bond in a September 2001 Treasury bond futures contract is a 13% coupon bond, and delivery is expected to be made on September 30, 2001. Coupon payments on the bond are made on February 4 and August 4 each year. The term structure is flat, and the rate of interest with semiannual compounding is 12% per annum. The conversion factor for the bond is 1.5. The current quoted bond price is \$110. Calculate the quoted futures price for the contract.
- 5.19. An investor is looking for arbitrage opportunities in the Treasury bond futures market. What complications are created by the fact that the party with a short position can choose to deliver any bond with a maturity of over 15 years?

- 5.20. Assuming that zero rates are as in Problem 5.13, what is the value of an FRA that enables the holder to earn 9.5% for a three-month period starting in one year on a principal of \$1,000,000? The interest rate is expressed with quarterly compounding.
- 5.21. Suppose that the nine-month LIBOR interest rate is 8% per annum and the six-month LIBOR interest rate is 7.5% per annum (both with continuous compounding). Estimate the three-month Eurodollar futures price quote for a contract maturing in six months.
- 5.22. A five-year bond with a yield of 11% (continuously compounded) pays an 8% coupon at the end of each year.
- What is the bond's price?
  - What is the bond's duration?
  - Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.
  - Recalculate the bond's price on the basis of a 10.8% per annum yield and verify that the result is in agreement with your answer to (c).
- 5.23. Suppose that a bond portfolio with a duration of 12 years is hedged using a futures contract in which the underlying asset has a duration of four years. What is likely to be the impact on the hedge of the fact that the 12-year rate is less volatile than the four-year rate?
- 5.24. Suppose that it is February 20 and a treasurer realizes that on July 17 the company will have to issue \$5 million of commercial paper with a maturity of 180 days. If the paper were issued today, the company would realize \$4,820,000. (In other words, the company would receive \$4,820,000 for its paper and have to redeem it at \$5,000,000 in 180 days' time.) The September Eurodollar futures price is quoted as 92.00. How should the treasurer hedge the company's exposure?
- 5.25. On August 1 a portfolio manager has a bond portfolio worth \$10 million. The duration of the portfolio in October will be 7.1 years. The December Treasury bond futures price is currently 91-12 and the cheapest-to-deliver bond will have a duration of 8.8 years at maturity. How should the portfolio manager immunize the portfolio against changes in interest rates over the next two months?
- 5.26. How can the portfolio manager change the duration of the portfolio to 3.0 years in Problem 5.25?
- 5.27. Between February 28, 2002, and March 1, 2002, you have a choice between owning a government bond paying a 10% coupon and a corporate bond paying a 10% coupon. Consider carefully the day count conventions discussed in this chapter and decide which of the two bonds you would prefer to own. Ignore the risk of default.
- 5.28. Suppose that a Eurodollar futures quote is 88 for a contract maturing in 60 days. What is the LIBOR forward rate for the 60- to 150-day period? Ignore the difference between futures and forwards for the purposes of this question.
- 5.29. "When the zero curve is upward sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward sloping the reverse is true." Explain why this is so.
- 5.30. The three-month Eurodollar futures price for a contract maturing in six years is quoted as 95.20. The standard deviation of the change in the short-term interest rate in one year is 1.1%. Estimate the forward LIBOR interest rate for the period between 6.00 and 6.25 years in the future.

## Assignment Questions

- 5.31. Assume that a bank can borrow or lend money at the same interest rate in the LIBOR market. The 91-day rate is 10% per annum, and the 182-day rate is 10.2% per annum, both expressed with continuous compounding. The Eurodollar futures price for a contract maturing in 91 days is quoted as 89.5. What arbitrage opportunities are open to the bank?
- 5.32. A Canadian company wishes to create a Canadian LIBOR futures contract from a U.S. Eurodollar futures contract and forward contracts on foreign exchange. Using an example, explain how the company should proceed. For the purposes of this problem, assume that a futures contract is the same as a forward contract.
- 5.33. Portfolio A consists of a one-year zero-coupon bond with a face value of \$2,000 and a 10-year zero-coupon bond with a face value of \$6,000. Portfolio B consists of a 5.95-year discount bond with a face value of \$5,000. The current yield on all bonds is 10% per annum.
- Show that both portfolios have the same duration.
  - Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.
  - What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?
- 5.34. The following table gives the prices of bonds:

Bond principal (\$)	Time to maturity (years)	Annual coupon (\$)*	Bond price (\$)
100	0.50	0.0	98
100	1.00	0.0	95
100	1.50	6.2	101
100	2.00	8.0	104

\* Half the stated coupon is assumed to be paid every six months.

- Calculate zero rates for maturities of 6 months, 12 months, 18 months, and 24 months.
  - What are the forward rates for the periods: 6 months to 12 months, 12 months to 18 months, 18 months to 24 months?
  - What are the 6-month, 12-month, 18-month, and 24-month par yields for bonds that provide semiannual coupon payments?
  - Estimate the price and yield of a two-year bond providing a semiannual coupon of 7% per annum.
- 5.35. It is June 25, 2001. The futures price for the June 2001 CBOT bond futures contract is 118-23.
- Calculate the conversion factor for a bond maturing on January 1, 2017, paying a coupon of 10%.
  - Calculate the conversion factor for a bond maturing on October 1, 2022, paying a coupon of 7%.
  - Suppose that the quoted prices of the bonds in (a) and (b) are 169.00 and 136.00, respectively. Which bond is cheaper to deliver?
  - Assuming that the cheapest to deliver bond is actually delivered, what is the cash price received for the bond?

- 5.36. A portfolio manager plans to use a Treasury bond futures contract to hedge a bond portfolio over the next three months. The portfolio is worth \$100 million and will have a duration of 4.0 years in three months. The futures price is 122, and each futures contract is on \$100,000 of bonds. The bond that is expected to be cheapest to deliver will have a duration of 9.0 years at the maturity of the futures contract. What position in futures contracts is required?
- What adjustments to the hedge are necessary if after one month the bond that is expected to be cheapest to deliver changes to one with a duration of seven years?
  - Suppose that all rates increase over the three months, but long-term rates increase less than short-term and medium-term rates. What is the effect of this on the performance of the hedge?