

# CHAPTER

# Properties of Stock Options

In this chapter we look at the factors affecting stock option prices. We use a number of different arbitrage arguments to explore the relationships between European option prices, American option prices, and the underlying stock price. The most important of these relationships is put–call parity, which is a relationship between European call option prices and European put option prices.

The chapter examines whether American options should be exercised early. It shows that it is never optimal to exercise an American call option on a non-dividend-paying stock prior to the option's expiration, but that under some circumstances the early exercise of an American put option on such a stock is optimal.

## 8.1 FACTORS AFFECTING OPTION PRICES

There are six factors affecting the price of a stock option:

1. The current stock price,  $S_0$
2. The strike price,  $X$
3. The time to expiration,  $T$
4. The volatility of the stock price,  $\sigma$
5. The risk-free interest rate,  $r$
6. The dividends expected during the life of the option

In this section we consider what happens to option prices when one of these factors changes with all the others remaining fixed. The results are summarized in Table 8.1.

Figures 8.1 and 8.2 show how the price of a European call and put depends on the first five factors in the situation where  $S_0 = 50$ ,  $X = 50$ ,  $r = 5\%$  per annum,  $\sigma = 30\%$  per annum,  $T = 1$  year, and there are no dividends. In this case the call price is 7.116 and the put price is 4.677.

### Stock Price and Strike Price

If a call option is exercised at some future time, the payoff will be the amount by which the stock price exceeds the strike price. Call options therefore become more valuable as

**Table 8.1** Summary of the effect on the price of a stock option of increasing one variable while keeping all others fixed\*

Variable	European call	European put	American call	American put
Current stock price	+	-	+	-
Strike price	-	+	-	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	-	+	-
Dividends	-	+	-	+

\* + indicates that an increase in the variable causes the option price to increase; - indicates that an increase in the variable causes the option price to decrease; ? indicates that the relationship is uncertain.

the stock price increases and less valuable as the strike price increases. For a put option, the payoff on exercise is the amount by which the strike price exceeds the stock price. Put options, therefore, behave in the opposite way from call options. They become less valuable as the stock price increases and more valuable as the strike price increases. Figures 8.1a, b, c, and d illustrate the way in which put and call prices depend on the stock price and strike price.

### Time to Expiration

Consider next the effect of the expiration date. Both put and call American options become more valuable as the time to expiration increases. Consider two options that differ only as far as the expiration date is concerned. The owner of the long-life option has all the exercise opportunities open to the owner of the short-life option—and more. The long-life option must therefore always be worth at least as much as the short-life option. Figures 8.1e and f illustrate the way in which calls and puts depend on the time to expiration.

Although European put and call options usually become more valuable as the time to expiration increases, this is not always the case. Consider two European call options on a stock: one with an expiration date in one month, the other with an expiration date in two months. Suppose that a very large dividend is expected in six weeks. The dividend will cause the stock price to decline, so that the short-life option could be worth more than the long-life option.

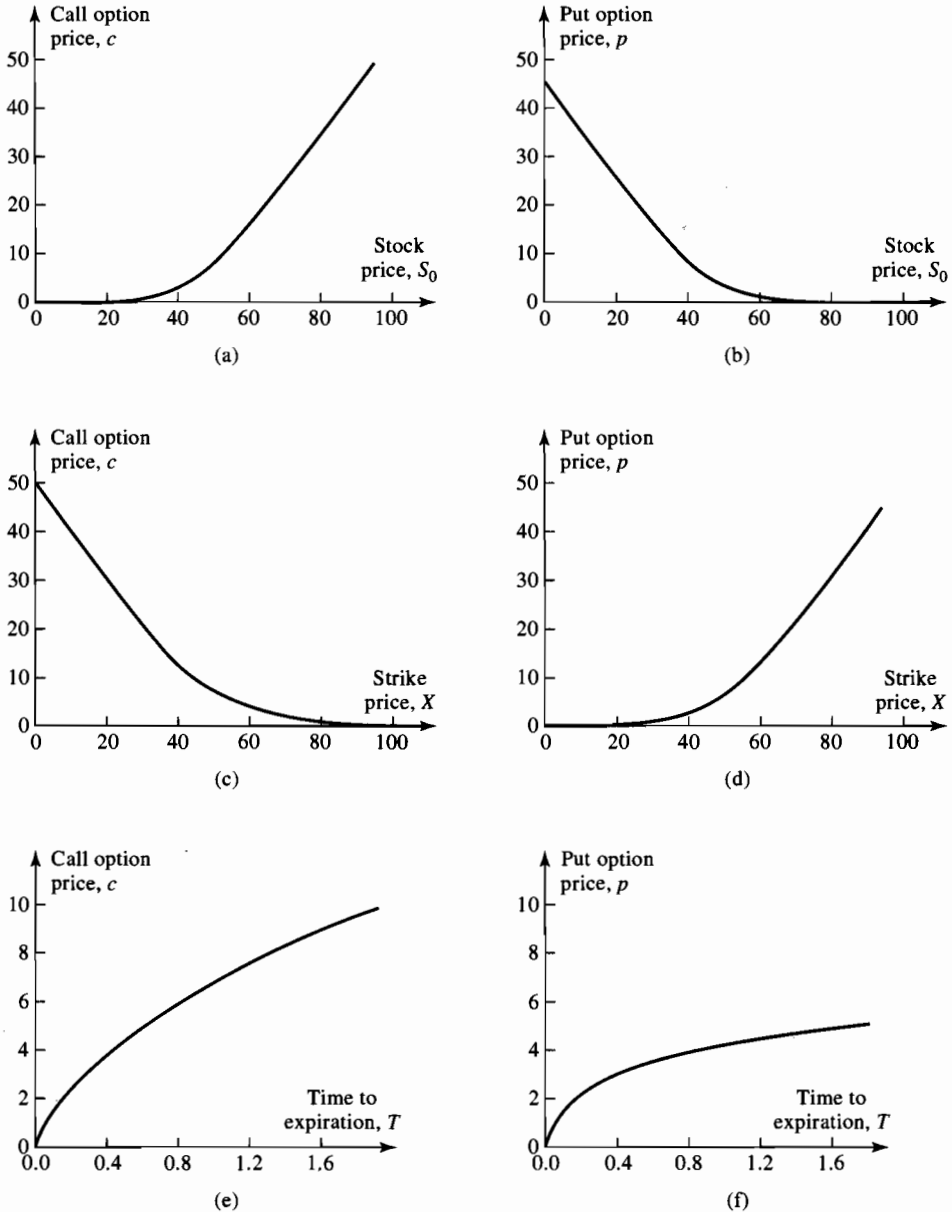
### Volatility

The precise way in which volatility is defined is discussed in Chapter 11. Roughly speaking, the *volatility* of a stock price is a measure of how uncertain we are about future stock price movements. As volatility increases, the chance that the stock will do very well or very poorly increases. For the owner of a stock, these two outcomes tend to offset each other. However, this is not so for the owner of a call or put. The owner of a call benefits from price increases but has limited downside risk in the event of price decreases because the most the owner can lose is the price of the option. Similarly, the owner of a put benefits from price decreases, but has limited downside risk in the event

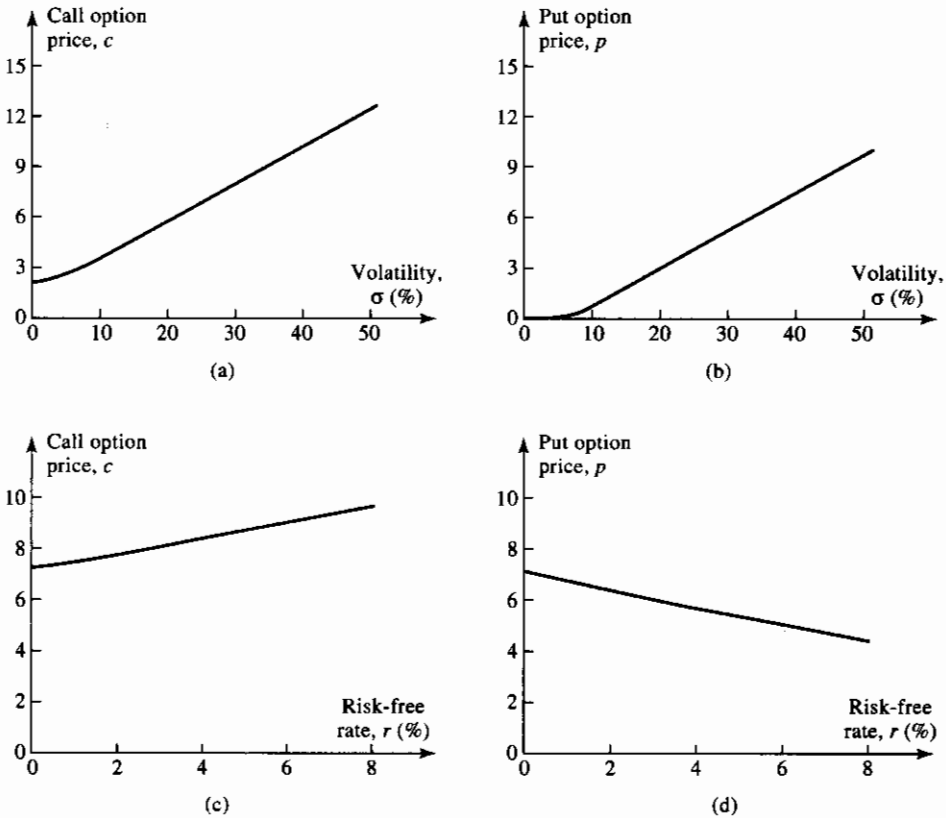
of price increases. The values of both calls and puts therefore increase as volatility increases (see Figures 8.2a and b).

### Risk-Free Interest Rate

The risk-free interest rate affects the price of an option in a less clear-cut way. As interest rates in the economy increase, the expected growth rate of the stock price tends to increase. However, the present value of a future cash flow received by the holder of



**Figure 8.1** Effect of changes in stock price, strike price, and expiration date on option prices  $S_0 = 50$ ,  $X = 50$ ,  $r = 5\%$ ,  $\sigma = 30\%$ , and  $T = 1$



**Figure 8.2** Effect of changes in volatility and risk-free interest rate on option prices when  $S_0 = 50$ ,  $X = 50$ ,  $r = 5\%$ ,  $\sigma = 30\%$ , and  $T = 1$

the option decreases. These two effects both tend to decrease the value of a put option. Hence, put option prices decline as the risk-free interest rate increases (see Figure 8.2d). In the case of calls, the first effect tends to increase the price, whereas the second effect tends to decrease it. It can be shown that the first effect always dominates the second effect; that is, the prices of calls always increase as the risk-free interest rate increases (see Figure 8.2c).

It must be emphasized that the results in this section assume that one of the six variables change while the other five variables remain fixed. In practice, when interest rates rise (fall), stock prices tend to fall (rise). The net effect of an interest rate increase and the accompanying stock price decrease can be to decrease the value of a call option and increase the value of a put option. Similarly, the net effect of an interest rate decrease and the accompanying stock price increase can be to increase the value of a call option and decrease the value of a put option.

**Dividends**

Dividends have the effect of reducing the stock price on the ex-dividend date. This is bad news for the value of call options and good news for the value of put options. The value of a call option is therefore negatively related to the size of any anticipated dividends, and the value of a put option is positively related to the size of any anticipated dividends.

## 8.2 ASSUMPTIONS AND NOTATION

In this chapter we will make assumptions similar to those made for deriving forward and futures prices in Chapter 3. We assume that there are some market participants, such as large investment banks, for which

1. There are no transactions costs.
2. All trading profits (net of trading losses) are subject to the same tax rate.
3. Borrowing and lending are possible at the risk-free interest rate.

We assume that these market participants are prepared to take advantage of arbitrage opportunities as they arise. As discussed in Chapters 1 and 3, this means that any available arbitrage opportunities disappear very quickly. For the purposes of our analyses, it is therefore reasonable to assume that there are no arbitrage opportunities.

We will use the following notation:

- $S_0$ : Current stock price
- $X$ : Strike price of option
- $T$ : Time to expiration of option
- $S_T$ : Stock price at option maturity
- $r$ : Continuously compounded risk-free rate of interest for an investment maturing in time  $T$
- $C$ : Value of American call option to buy one share
- $P$ : Value of American put option to sell one share
- $c$ : Value of European call option to buy one share
- $p$ : Value of European put option to sell one share

It should be noted that  $r$  is the nominal rate of interest, not the real rate of interest. We can assume that  $r > 0$ . Otherwise, a risk-free investment would provide no advantages over cash. (Indeed, if  $r < 0$ , cash would be preferable to a risk-free investment.)

## 8.3 UPPER AND LOWER BOUNDS FOR OPTION PRICES

In this section we derive upper and lower bounds for option prices. These bounds do not depend on any particular assumptions about the factors mentioned in the previous section (except  $r > 0$ ). If an option price is above the upper bound or below the lower bound, there are profitable opportunities for arbitrageurs.

### Upper Bounds

An American or European call option gives the holder the right to buy one share of a stock for a certain price. No matter what happens, the option can never be worth more than the stock. Hence, the stock price is an upper bound to the option price:

$$c \leq S_0 \quad \text{and} \quad C \leq S_0$$

If these relationships were not true, an arbitrageur could easily make a riskless profit by buying the stock and selling the call option.

An American or European put option gives the holder the right to sell one share of a

stock for  $X$ . No matter how low the stock price becomes, the option can never be worth more than  $X$ . Hence,

$$p \leq X \quad \text{and} \quad P \leq X$$

For European options, we know that at maturity the option cannot be worth more than  $X$ . It follows that it cannot be worth more than the present value of  $X$  today:

$$p \leq Xe^{-rT}$$

If this were not true, an arbitrageur could make a riskless profit by writing the option and investing the proceeds of the sale at the risk-free interest rate.

### Lower Bound for Calls on Non-Dividend-Paying Stocks

A lower bound for the price of a European call option on a non-dividend-paying stock is

$$S_0 - Xe^{-rT}$$

We first look at a numerical example and then consider a more formal argument.

Suppose that  $S_0 = \$20$ ,  $X = \$18$ ,  $r = 10\%$  per annum, and  $T = 1$  year. In this case,

$$S_0 - Xe^{-rT} = 20 - 18e^{-0.1} = 3.71$$

or \$3.71. Consider the situation where the European call price is \$3.00, which is less than the theoretical minimum of \$3.71. An arbitrageur can buy the call and short the stock to provide a cash inflow of  $\$20.00 - \$3.00 = \$17.00$ . If invested for one year at 10% per annum, the \$17.00 grows to  $17e^{0.1} = \$18.79$ . At the end of the year, the option expires. If the stock price is greater than \$18.00, the arbitrageur exercises the option for \$18.00, closes out the short position, and makes a profit of

$$\$18.79 - \$18.00 = \$0.79$$

If the stock price is less than \$18.00, the stock is bought in market and the short position is closed out. The arbitrageur then makes an even greater profit. For example, if the stock price is \$17.00, the arbitrageur's profit is

$$\$18.79 - \$17.00 = \$1.79$$

This example is illustrated in Table 8.2.

For a more formal argument, we consider the following two portfolios:

*Portfolio A:* one European call option plus an amount of cash equal to  $Xe^{-rT}$

*Portfolio B:* one share

In portfolio A, the cash, if it is invested at the risk-free interest rate, will grow to  $X$  in time  $T$ . If  $S_T > X$ , the call option is exercised at maturity and portfolio A is worth  $S_T$ . If  $S_T < X$ , the call option expires worthless and the portfolio is worth  $X$ . Hence, in time  $T$  portfolio A is worth

$$\max(S_T, X)$$

Portfolio B is worth  $S_T$  in time  $T$ . Hence, portfolio A is always worth as much as, and

**Table 8.2** Arbitrage opportunity when European call price is less than the lower bound*From the Trader's Desk*

An investor has just obtained the following quotes for a European call option on a non-dividend-paying stock with a strike price of \$18 and an expiration date in one year:

Stock price: \$20

Option price: \$3

The risk-free interest rate for a one-year investment is 10% per annum.

*Opportunity*

1. Buy the option.
2. Short the stock.
3. Invest surplus cash at 10% per annum.

*The Result*

This strategy provides an immediate positive cash flow of  $\$20.00 - \$3.00 = \$17.00$ . The \$17.00 is invested at 10% per annum and grows to  $17e^{0.1} = \$18.79$  at the end of one year. At this time the option expires. If the price of the stock is greater than \$18.00, the investor exercises the option and closes out the short position for a profit of

$$\$18.79 - \$18.00 = \$0.79$$

If the price of the stock is less than \$18.00 at the end of one year, the stock is bought in the market and the short position is closed out. The investor then makes a profit equal to

$$18.79 - S_T$$

where  $S_T$  is the stock price. Because  $S_T < 18$ , the profit is at least as great as \$0.79.

can be worth more than, portfolio B at the option's maturity. It follows that in the absence of arbitrage opportunities this must also be true today. Hence,

$$c + Xe^{-rT} \geq S_0$$

or

$$c \geq S_0 - Xe^{-rT}$$

Because the worst that can happen to a call option is that it expires worthless, its value cannot be negative. This means that  $c \geq 0$  and therefore

$$c \geq \max(S_0 - Xe^{-rT}, 0) \quad (8.1)$$

**Example**

Consider a European call option on a non-dividend-paying stock when the stock price is \$51, the exercise price is \$50, the time to maturity is six months, and the risk-free rate of interest is 12% per annum. In this case,  $S_0 = 51$ ,  $X = 50$ ,  $T = 0.5$ , and  $r = 0.12$ . From equation (8.1), a lower bound for the option price is  $S_0 - Xe^{-rT}$ , or

$$51 - 50e^{-0.12 \times 0.5} = \$3.91$$

**Lower Bound for European Puts on Non-Dividend-Paying Stocks**

For a European put option on a non-dividend-paying stock, a lower bound for the price is

$$Xe^{-rT} - S_0$$

Again, we first consider a numerical example and then look at a more formal argument.

Suppose that  $S_0 = \$37$ ,  $X = \$40$ ,  $r = 5\%$  per annum, and  $T = 0.5$  years. In this case

$$Xe^{-rT} - S_0 = 40e^{-0.05 \times 0.5} - 37 = \$2.01$$

Consider the situation where the European put price is \$1.00, which is less than the theoretical minimum of \$2.01. An arbitrageur can borrow \$38.00 for six months to buy both the put and the stock. At the end of the six months, the arbitrageur will be required to repay  $38e^{0.05 \times 0.5} = \$38.96$ . If the stock price is below \$40.00, the arbitrageur exercises the option to sell the stock for \$40.00, repays the loan, and makes a profit of

$$\$40.00 - \$38.96 = \$1.04$$

If the stock price is greater than \$40.00, the arbitrageur discards the option, sells the stock, and repays the loan for an even greater profit. For example, if the stock price is \$42.00, the arbitrageur's profit is

$$\$42.00 - \$38.96 = \$3.04$$

This example is illustrated in Table 8.3.

For a more formal argument, we consider the following two portfolios:

*Portfolio C:* one European put option plus one share

*Portfolio D:* an amount of cash equal to  $Xe^{-rT}$

If  $S_T < X$ , the option in portfolio C is exercised at the option maturity, and the portfolio becomes worth  $X$ . If  $S_T > X$ , the put option expires worthless, and the portfolio is worth  $S_T$  at this time. Hence, portfolio C is worth

$$\max(S_T, X)$$

in time  $T$ . Assuming the cash is invested at the risk-free interest rate, portfolio D is worth  $X$  in time  $T$ . Hence, portfolio C is always worth as much as, and can sometimes be worth more than, portfolio D in time  $T$ . It follows that in the absence of arbitrage opportunities portfolio C must be worth at least as much as portfolio D today. Hence,

$$p + S_0 \geq Xe^{-rT}$$

or

$$p \geq Xe^{-rT} - S_0$$

Because the worst that can happen to a put option is that it expires worthless, its value cannot be negative. This means that

$$p \geq \max(Xe^{-rT} - S_0, 0) \tag{8.2}$$

**Table 8.3** Arbitrage opportunities when European put price is less than the lower bound*From the Trader's Desk*

An investor has just obtained the following quotes for a European put option on a non-dividend-paying stock with a strike price of \$40 and an expiration date in six months.

Stock price: \$37

Option price: \$1

The risk-free interest rate for a six-month investment is 5% per annum.

*Opportunity*

1. Borrow \$38 for six months.
2. Buy one option.
3. Buy one share of the stock.

*The Result*

At the end of the six months,  $38e^{0.05 \times 6/12} = \$38.96$  is required to pay off the loan. If the stock price at this time is below \$40.00, the investor exercises the option to sell the stock for \$40.00 and makes a profit of

$$\$40.00 - \$38.96 = \$1.04$$

If the price of the stock is greater than \$40.00, the investor sells the stock and repays the loan for a profit of

$$S_T - 38.96$$

where  $S_T$  is the stock price. The profit is at least as great as \$1.04.

**Example**

Consider a European put option on a non-dividend-paying stock when the stock price is \$38, the exercise price is \$40, the time to maturity is three months, and the risk-free rate of interest is 10% per annum. In this case  $S_0 = 38$ ,  $X = 40$ ,  $T = 0.25$ , and  $r = 0.10$ . From equation (8.2), a lower bound for the option price is  $Xe^{-rT} - S_0$ , or

$$40e^{-0.1 \times 0.25} - 38 = \$1.01$$

**8.4 PUT-CALL PARITY**

We now derive an important relationship between  $p$  and  $c$ . Consider the following two portfolios that were used in the previous section:

*Portfolio A:* one European call option plus an amount of cash equal to  $Xe^{-rT}$

*Portfolio C:* one European put option plus one share

Both are worth

$$\max(S_T, X)$$

at expiration of the options. Because the options are European, they cannot be exercised prior to the expiration date. The portfolios must therefore have identical

**Table 8.4** Arbitrage opportunity when put–call parity does not hold: call price too low relative to put price*From the Trader's Desk*

An investor has just obtained the following quotes for options on a stock worth \$31 when the three-month risk-free interest rate is 10% per annum. Both options have a strike price of \$30 and an expiration date in three months.

European call: \$3

European put: \$2.25

*The Strategy*

1. Buy the call.
2. Short the put.
3. Short the stock.

*The Outcome*

This strategy leads to an initial cash flow of  $\$31.00 - \$3.00 + \$2.25 = \$30.25$ . When invested for three months at the risk-free interest rate, this amount grows to  $30.25e^{0.1 \times 3/12} = \$31.02$ . At the end of the three months, the possible situations are as follows:

1. Stock price is greater than \$30.00. The investor exercises the call. This involves buying one share for \$30.00. The short position is closed out, and the net profit is  $\$31.02 - \$30.00 = \$1.02$ .
2. Stock price is less than \$30.00. The counterparty exercises the put. This also involves the investor in buying one share for \$30.00. The short position is closed out, and the net profit is again  $\$31.02 - \$30.00 = \$1.02$ .

values today. This means that

$$c + Xe^{-rT} = p + S_0 \quad (8.3)$$

This relationship is known as *put–call parity*. It shows that the value of a European call with a certain exercise price and exercise date can be deduced from the value of a European put with the same exercise price and exercise date, and vice versa.

If equation (8.3) does not hold, there are arbitrage opportunities. Suppose that the stock price is \$31, the exercise price is \$30, the risk-free interest rate is 10% per annum, the price of a three-month European call option is \$3, and the price of a three-month European put option is \$2.25. In this case

$$c + Xe^{-rT} = 3 + 30e^{-0.1 \times 3/12} = \$32.26$$

$$p + S_0 = 2.25 + 31 = \$33.25$$

Portfolio C is overpriced relative to portfolio A. The correct arbitrage strategy is to buy the securities in portfolio A and short the securities in portfolio C. The strategy involves buying the call and shorting both the put and the stock, generating a positive cash flow of

$$-3 + 2.25 + 31 = \$30.25$$

up front. When invested at the risk-free interest rate, this amount grows to  $30.25e^{0.1 \times 0.25} = \$31.02$  in three months.

If the stock price at expiration of the option is greater than \$30, the call will be

**Table 8.5** Arbitrage opportunity when put-call parity does not hold: put price too low relative to call price

*From the Trader's Desk*

An investor has just obtained the following quotes for options on a stock worth \$31 when the three-month risk-free interest rate is 10% per annum. Both options have a strike price of \$30 and an expiration date in three months.

European call: \$3

European put: \$1

*The Strategy*

1. Sell the call.
2. Buy the put.
3. Buy the stock.

*The Outcome*

This strategy involves an investment of  $\$31 + \$1 - \$3 = \$29$  at time zero. When the investment is financed at the risk-free interest rate, a repayment of  $29e^{0.1 \times 3/12} = \$29.73$  is required at the end of three months. The possible situations are as follows:

1. Stock price is greater than \$30.00. The counterparty exercises the call. This means that the investor has to sell the share owned for \$30.00. The net profit is  $\$30.00 - \$29.73 = \$0.27$ .
2. Stock price is less than \$30.00. The investor exercises the put. This also means that the share is sold for \$30.00. The net profit is again  $\$30.00 - \$29.73 = \$0.27$ .

exercised. If it is less than \$30, the put will be exercised. In either case, the investor ends up buying one share for \$30. This share can be used to close out the short position. The net profit is therefore

$$\$31.02 - \$30.00 = \$1.02$$

This example is illustrated in Table 8.4.

For an alternative situation, suppose that the call price is \$3 and the put price is \$1. In this case

$$c + Xe^{-rT} = 3 + 30e^{-0.1 \times 3/12} = \$32.26$$

$$p + S_0 = 1 + 31 = \$32.00$$

Portfolio A is overpriced relative to portfolio C. An arbitrageur can short the securities in portfolio A and buy the securities in portfolio C to lock in a profit. The strategy involves shorting the call and buying both the put and the stock with an initial investment of

$$\$31 + \$1 - \$3 = \$29$$

When the investment is financed at the risk-free interest rate, a repayment of  $29e^{0.1 \times 0.25} = \$29.73$  is required at the end of the three months. As in the previous case, either the call or the put will be exercised. The short call and long put option position therefore leads to the stock being sold for \$30.00. The net profit is therefore

$$\$30.00 - \$29.73 = \$0.27$$

This example is illustrated in Table 8.5.

### American Options

Put-call parity holds only for European options. However, it is possible to derive some results for American option prices. It can be shown (see Problem 8.17) that

$$S_0 - X \leq C - P \leq S_0 - Xe^{-rT} \quad (8.4)$$

#### Example

An American call option on a non-dividend-paying stock with exercise price \$20.00 and maturity in five months is worth \$1.50. Suppose that the current stock price is \$19.00 and the risk-free interest rate is 10% per annum. From equation (8.4)

$$19 - 20 \leq C - P \leq 19 - 20e^{-0.1 \times 5/12}$$

or

$$1 \geq P - C \geq 0.18$$

showing that  $P - C$  lies between \$1.00 and \$0.18. With  $C$  at \$1.50,  $P$  must lie between \$1.68 and \$2.50. In other words, upper and lower bounds for the price of an American put with the same strike price and expiration date as the American call are \$2.50 and \$1.68.

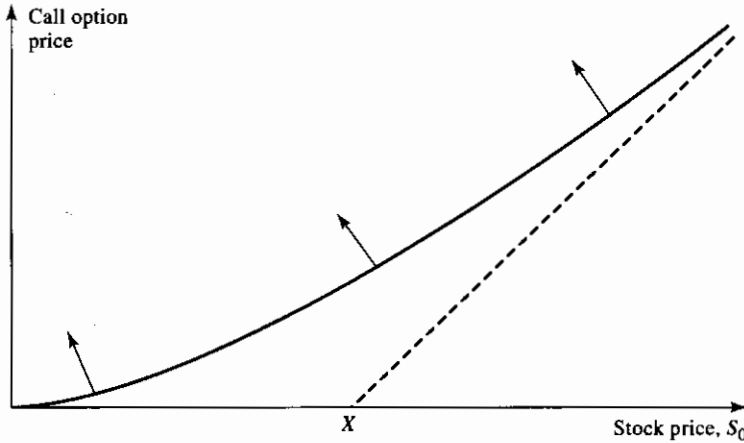
## 8.5 EARLY EXERCISE: CALLS ON A NON-DIVIDEND-PAYING STOCK

This section demonstrates that it is never optimal to exercise an American call option on a non-dividend-paying stock before the expiration date.

To illustrate the general nature of the argument, consider an American call option on a non-dividend-paying stock with one month to expiration when the stock price is \$50 and the strike price is \$40. The option is deep in the money, and the investor who owns the option might well be tempted to exercise it immediately. However, if the investor plans to hold the stock for more than one month, this is not the best strategy. A better course of action is to keep the option and exercise it at the end of the month. The \$40 strike price is then paid out one month later than it would be if the option were exercised immediately, so that interest is earned on the \$40 for one month. Because the stock pays no dividends, no income from the stock is sacrificed. A further advantage of waiting rather than exercising immediately is that there is some chance (however remote) that the stock price will fall below \$40 in one month. In this case the investor will not exercise and will be glad that the decision to exercise early was not taken!

This argument shows that there are no advantages to exercising early if the investor plans to keep the stock for the remaining life of the option (one month, in this case). What if the investor thinks the stock is currently overpriced and is wondering whether to exercise the option and sell the stock? In this case, the investor is better off selling the option than exercising it.<sup>1</sup> The option will be bought by another investor who does want to hold the stock. Such investors must exist. Otherwise the current stock price would not be \$50. The price obtained for the option will be greater than its intrinsic value of \$10 for the reasons mentioned earlier.

<sup>1</sup> As an alternative strategy, the investor can keep the option and short the stock to lock in a better profit than \$10.



**Figure 8.3** Variation of price of an American or European call option on a non-dividend-paying stock with the stock price,  $S_0$

For a more formal argument, we can use equation (8.1):

$$c \geq S_0 - Xe^{-rT}$$

Because the owner of an American call has all the exercise opportunities open to the owner of the corresponding European call, we must have

$$C \geq c$$

Hence,

$$C \geq S_0 - Xe^{-rT}$$

Given  $r > 0$ , it follows that  $C > S_0 - X$ . If it were optimal to exercise early,  $C$  would equal  $S_0 - X$ . We deduce that it can never be optimal to exercise early.

Figure 8.3 shows the general way in which the call price varies with  $S_0$  and  $X$ . It indicates that the call price is always above its intrinsic value of  $\max(S_0 - X, 0)$ . As  $r$  or  $T$  or the volatility increases, the line relating the call price to the stock price moves in the direction indicated by the arrows (i.e., farther away from the intrinsic value).

To summarize, there are two reasons an American call on a non-dividend-paying stock should not be exercised early. One relates to the insurance that it provides. A call option, when held instead of the stock itself, in effect insures the holder against the stock price falling below the exercise price. Once the option has been exercised and the exercise price has been exchanged for the stock price, this insurance vanishes. The other reason concerns the time value of money. From the perspective of the option holder, the later the strike price is paid out the better.

## 8.6 EARLY EXERCISE: PUTS ON A NON-DIVIDEND-PAYING STOCK

It can be optimal to exercise an American put option on a non-dividend-paying stock early. Indeed, at any given time during its life, a put option should always be exercised early if it is sufficiently deeply in the money.

To illustrate, consider an extreme situation. Suppose that the strike price is \$10 and the stock price is virtually zero. By exercising immediately, an investor makes an

immediate gain of \$10. If the investor waits, the gain from exercise might be less than \$10 but it cannot be more than \$10, because negative stock prices are impossible. Furthermore, receiving \$10 now is preferable to receiving \$10 in the future. It follows that the option should be exercised immediately.

Like a call option, a put option can be viewed as providing insurance. A put option, when held in conjunction with the stock, insures the holder against the stock price falling below a certain level. However, a put option is different from a call option in that it may be optimal for an investor to forgo this insurance and exercise early in order to realize the strike price immediately. In general, the early exercise of a put option becomes more attractive as  $S_0$  decreases, as  $r$  increases, and as the volatility decreases.

It will be recalled from equation (8.2) that

$$p \geq X e^{-rT} - S_0$$

For an American put with price  $P$ , the stronger condition

$$P \geq X - S_0$$

must always hold because immediate exercise is always possible.

Figure 8.4 shows the general way in which the price of an American put varies with  $S_0$ . Provided that  $r > 0$ , it is always optimal to exercise an American put immediately when the stock price is sufficiently low. When early exercise is optimal, the value of the option is  $X - S_0$ . The curve representing the value of the put therefore merges into the put's intrinsic value,  $X - S_0$ , for a sufficiently small value of  $S_0$ . In Figure 8.4, this value of  $S_0$  is shown as point A. The line relating the put price to the stock price moves in the direction indicated by the arrows when  $r$  decreases, when the volatility increases, and when  $T$  increases.

Because there are some circumstances when it is desirable to exercise an American put option early, it follows that an American put option is always worth more than the corresponding European put option. Furthermore, because an American put is sometimes worth its intrinsic value (see Figure 8.4), it follows that a European put option must sometimes be worth less than its intrinsic value. Figure 8.5 shows the variation of the European put price with the stock price. Note that point B in Figure 8.5, at which

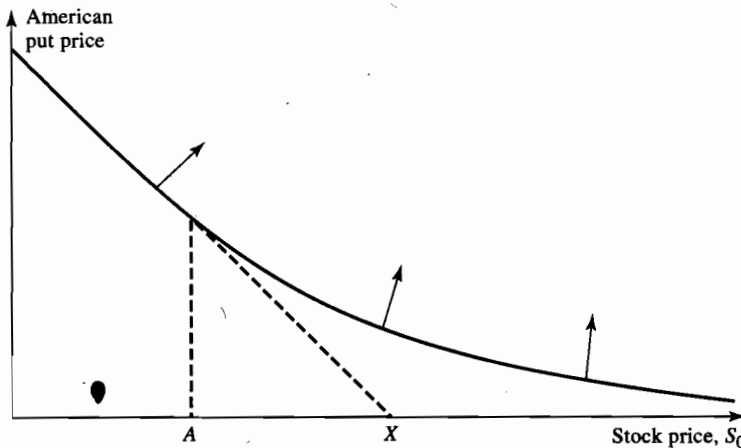
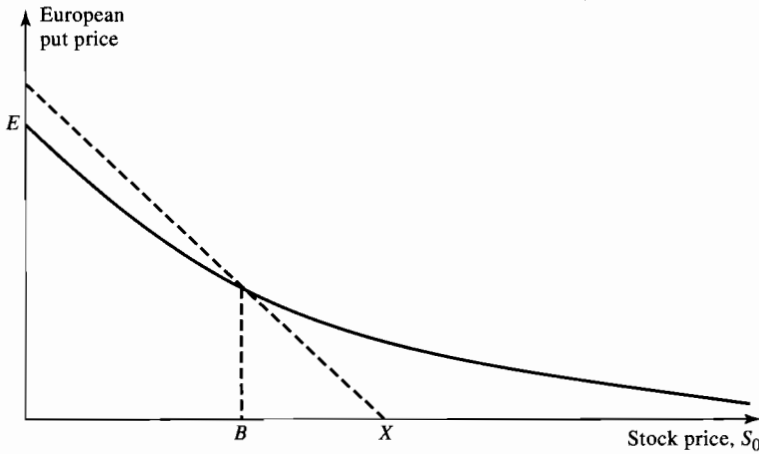


Figure 8.4 Variation of price of an American put option with stock price,  $S_0$



**Figure 8.5** Variation of price of a European put option with the stock price,  $S_0$

the price of the option is equal to its intrinsic value, must represent a higher value of the stock price than point A in Figure 8.4. Point E in Figure 8.5 is where  $S_0 = 0$  and the European put price is  $Xe^{-rT}$ .

## 8.7 EFFECT OF DIVIDENDS

The results produced so far in this chapter have assumed that we are dealing with options on a non-dividend-paying stock. In this section we examine the impact of dividends. In the United States exchange-traded stock options generally have less than eight months to maturity. The dividends payable during the life of the option can usually be predicted with reasonable accuracy. We will use  $D$  to denote the present value of the dividends during the life of the option. In the calculation of  $D$ , a dividend is assumed to occur at the time of its ex-dividend date.

### Lower Bound for Calls and Puts

We can redefine portfolios A and B as follows:

*Portfolio A:* one European call option plus an amount of cash equal to  $D + Xe^{-rT}$

*Portfolio B:* one share

A similar argument to the one used to derive equation (8.1) shows that

$$c \geq S_0 - D - Xe^{-rT} \quad (8.5)$$

We can also redefine portfolios C and D as follows:

*Portfolio C:* one European put option plus one share

*Portfolio D:* an amount of cash equal to  $D + Xe^{-rT}$

A similar argument to the one used to derive equation (8.2) shows that

$$p \geq D + Xe^{-rT} - S_0 \quad (8.6)$$

### Early Exercise

When dividends are expected, we can no longer assert that an American call option will not be exercised early. Sometimes it is optimal to exercise an American call immediately prior to an ex-dividend date. It is never optimal to exercise a call at other times. This point is discussed further in Chapter 11.

### Put-Call Parity

Comparing the value at option maturity of the redefined portfolios A and C shows that, with dividends, the put-call parity result in equation (8.3) becomes

$$c + D + Xe^{-rT} = p + S_0 \quad (8.7)$$

Dividends cause equation (8.4) to be modified (see Problem 8.18) to

$$S_0 - D - X \leq C - P \leq S_0 - Xe^{-rT} \quad (8.8)$$

## 8.8 EMPIRICAL RESEARCH

Empirical research to test the results in this chapter might seem to be relatively simple to carry out once the appropriate data have been assembled. In fact, there are a number of complications:

1. It is important to be sure that option prices and stock prices are being observed at exactly the same time. For example, testing for arbitrage opportunities by looking at the price at which the last trade is done each day is inappropriate. This point was made in Chapter 7 in connection with the data in Table 7.3.
2. It is important to consider carefully whether a trader can take advantage of any observed arbitrage opportunity. If the opportunity exists only momentarily, there may in practice be no way of exploiting it.
3. Transactions costs must be taken into account when determining whether arbitrage opportunities are possible.
4. Put-call parity holds only for European options. Exchange-traded stock options are American.
5. Dividends to be paid during the life of the option must be estimated.

Some of the empirical research that has been carried out is described in the papers by Bhattacharya, Galai, Gould and Galai, Klemkosky and Resnick, and Stoll, all referenced at the end of this chapter. Galai and Bhattacharya test whether option prices are ever less than their theoretical lower bounds; Stoll, Gould and Galai, and the two papers by Klemkosky and Resnick test whether put-call parity holds. We will consider the results of Bhattacharya and Klemkosky and Resnick.

Bhattacharya's study examined whether the theoretical lower bounds for call options applied in practice. He used data consisting of the transactions prices for options on 58 stocks over a 196-day period between August 1976 and June 1977. The first test examined whether the options satisfied the condition that price be at least as great as the intrinsic value—that is, whether  $C \geq \max(S_0 - X, 0)$ . More than 86,000 option prices were examined and about 1.3% were found to violate this condition. In 29% of the cases, the violation disappeared by the next trade, indicating that in practice traders

would not have been able to take advantage of it. When transactions costs were taken into account, the profitable opportunities created by the violation disappeared. Bhattacharya's second test examined whether options sold for less than the lower bound  $S_0 - D - Xe^{-rT}$ . (See equation 8.5.) He found that 7.6% of his observations did in fact sell for less than this lower bound. However, when transactions costs were taken into account, these did not give rise to profitable opportunities.

Klemkosky and Resnick's tests of put-call parity used data on option prices taken from trades between July 1977 and June 1978. They subjected their data to several tests to determine the likelihood of options being exercised early, and they discarded data for which early exercise was considered probable. In doing so, they felt they were justified in treating American options as European. They identified 540 situations where an arbitrage opportunity similar to that in Table 8.4 existed and 540 situations where an arbitrage opportunity similar to that in Table 8.5 existed. After transactions costs were allowed for, 38 of the Table 8.4 opportunities (call price too low relative to put price) and 147 of the Table 8.5 opportunities (call price too high relative to the put price) were still profitable. The opportunities persisted when either a 5- or a 15-minute delay between the opportunity being noted and trades being executed was assumed. Klemkosky and Resnick's conclusion was that arbitrage opportunities were available to some traders, particularly market makers, during the period they studied.

## 8.9 SUMMARY

There are six factors affecting the value of a stock option: the current stock price, the strike price, the expiration date, the stock price volatility, the risk-free interest rate, and the dividends expected during the life of the option. The value of a call generally increases as the current stock price, the time to expiration, the volatility, and the risk-free interest rate increase. The value of a call decreases as the strike price and expected dividends increase. The value of a put generally increases as the strike price, the time to expiration, the volatility, and the expected dividends increase. The value of a put decreases as the current stock price and the risk-free interest rate increase.

It is possible to reach some conclusions about the value of stock options without making any assumptions about the volatility of stock prices. For example, the price of a call option on a stock must always be worth less than the price of the stock itself. Similarly, the price of a put option on a stock must always be worth less than the option's strike price.

A call option on a non-dividend-paying stock must be worth more than

$$\max(S_0 - Xe^{-rT}, 0)$$

where  $S_0$  is the stock price,  $X$  is the exercise price,  $r$  is the risk-free interest rate, and  $T$  is the time to expiration. A put option on a non-dividend-paying stock must be worth more than

$$\max(Xe^{-rT} - S_0, 0)$$

When dividends with present value  $D$  will be paid, the lower bound for a call option becomes

$$\max(S_0 - D - Xe^{-rT}, 0)$$

and the lower bound for a put option becomes

$$\max(Xe^{-rT} + D - S_0, 0)$$

Put-call parity is a relationship between the price,  $c$ , of a European call option on a stock and the price,  $p$ , of a European put option on a stock. For a non-dividend-paying stock, it is

$$c + Xe^{-rT} = p + S_0$$

For a dividend-paying stock, the put-call parity relationship is

$$c + D + Xe^{-rT} = p + S_0$$

Put-call parity does not hold for American options. However, it is possible to use arbitrage arguments to obtain upper and lower bounds for the difference between the price of an American call and the price of an American put.

In Chapter 11 we will carry the analyses in this chapter further by making specific assumptions about the probabilistic behavior of stock prices. The analysis will enable us to derive exact pricing formulas for European stock options. In Chapter 17, we will see how numerical procedures can be used to price American options.

## Suggestions for Further Reading

- Bhattacharya, M. "Transaction Data Tests of Efficiency of the Chicago Board Options Exchange." *Journal of Financial Economics* 12 (1983): 161–85.
- Galai, D. "Empirical Tests of Boundary Conditions for CBOE Options." *Journal of Financial Economics* 6 (1978): 187–211.
- Gould, J. P., and D. Galai. "Transactions Costs and the Relationship between Put and Call Prices." *Journal of Financial Economics* 1 (1974): 105–29.
- Klemkosky, R. C., and B. G. Resnick. "An Ex-Ante Analysis of Put-Call Parity." *Journal of Financial Economics* 8 (1980): 363–78.
- Klemkosky, R. C., and B. G. Resnick. "Put-Call Parity and Market Efficiency." *Journal of Finance* 34 (December 1979): 1141–155.
- Merton, R. C. "The Relationship between Put and Call Prices: Comment." *Journal of Finance* 28 (March 1973): 183–84.
- Merton, R. C. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (spring 1973): 141–83.
- Stoll, H. R. "The Relationship between Put and Call Option Prices." *Journal of Finance* 31 (May 1969): 319–32.

## Quiz (Answers at End of Book)

- 8.1. List the six factors affecting stock option prices.
- 8.2. What is a lower bound for the price of a four-month call option on a non-dividend-paying stock when the stock price is \$28, the strike price is \$25, and the risk-free interest rate is 8% per annum?
- 8.3. What is a lower bound for the price of a one-month European put option on a non-dividend-paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

- 8.4. Give two reasons that the early exercise of an American call option on a non-dividend-paying stock is not optimal. The first reason should involve the time value of money. The second reason should apply even if interest rates are zero.
- 8.5. "The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put." Explain this statement.
- 8.6. Explain why an American call option is always worth at least as much as its intrinsic value. Is the same true of a European call option? Explain your answer.
- 8.7. Explain why the arguments leading to put-call parity for European options cannot be used to give a similar result for American options.

### Questions and Problems (Answers in Solutions Manual)

- 8.8. What is a lower bound for the price of a six-month call option on a non-dividend-paying stock when the stock price is \$80, the strike price is \$75, and the risk-free interest rate is 10% per annum?
- 8.9. What is a lower bound for the price of a two-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk-free interest rate is 5% per annum?
- 8.10. A four-month European call option on a dividend-paying stock is currently selling for \$5. The stock price is \$64, the strike price is \$60, and a dividend of \$0.80 is expected in one month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur?
- 8.11. A one-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?
- 8.12. Give an intuitive explanation of why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.
- 8.13. The price of a European call that expires in six months and has a strike price of \$30 is \$2. The underlying stock price is \$29, and a dividend of \$0.50 is expected in two months and again in five months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put option that expires in six months and has a strike price of \$30?
- 8.14. Explain carefully the arbitrage opportunities in Problem 8.13 if the European put price is \$3.
- 8.15. The price of an American call on a non-dividend-paying stock is \$4. The stock price is \$31, the strike price is \$30, and the expiration date is in three months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.
- 8.16. Explain carefully the arbitrage opportunities in Problem 8.15 if the American put price is greater than the calculated upper bound.
- 8.17. Prove the result in equation (8.4). (*Hint:* For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to  $X$  and (b) a portfolio consisting of an American put option plus one share.)

- 8.18. Prove the result in equation (8.8). (*Hint:* For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to  $D + X$  and (b) a portfolio consisting of an American put option plus one share.)
- 8.19. Even when the company pays no dividends, there is a tendency for executive stock options to be exercised early. (See Section 7.12 for a discussion of executive stock options.) Give a possible reason for this.
- 8.20. Use the software DerivaGem to verify that Figures 8.1 and 8.2 are correct.

### Assignment Questions

- 8.21. A European call option and put option on a stock both have a strike price of \$20 and an expiration date in three months. Both sell for \$3. The risk-free interest rate is 10% per annum, the current stock price is \$19, and a \$1 dividend is expected in one month. Identify the arbitrage opportunity open to a trader.
- 8.22. Suppose that  $c_1$ ,  $c_2$ , and  $c_3$  are the prices of European call options with strike prices  $X_1$ ,  $X_2$ , and  $X_3$ , respectively, where  $X_3 > X_2 > X_1$  and  $X_3 - X_2 = X_2 - X_1$ . All options have the same maturity. Show that

$$c_2 \leq 0.5(c_1 + c_3)$$

(*Hint:* Consider a portfolio that is long one option with strike price  $X_1$ , long one option with strike price  $X_3$ , and short two options with strike price  $X_2$ .)

- 8.23. What is the result corresponding to that in Problem 8.22 for European put options?
- 8.24. Suppose that you are the manager and sole owner of a highly leveraged company. All the debt will mature in one year. If at that time the value of the company is greater than the face value of the debt, you will pay off the debt. If the value of the company is less than the face value of the debt, you will declare bankruptcy and the debt holders will own the company.
- Express your position as an option on the value of the company.
  - Express the position of the debt holders in terms of options on the value of the company.
  - What can you do to increase the value of your position?